AG Audretsch
Fachbereich Physik
Mathematisch-Naturwissenschaftliche Sektion
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Diplomarbeit

Sequential Quantum Measurements

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Gemäß der *Prüfungsordnung der Universität Konstanz für den Diplomstudiengang Physik* erkläre ich hiermit, dass ich die vorliegende Diplomarbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel verwendet habe.

Zusammenfassung auf Deutsch

Das Thema der vorliegenden Diplomarbeit sind sequenzielle quantenmechanische Messungen. In diesem Zusammenhang werden auch quantenmechanische Einzelmessungen betrachtet. Worum es sich bei dieser Art von Messungen handelt, wird zusammen mit weiteren Postulaten der Quantenmechanik zu Beginn von Kapitel 1 definiert. Dabei wird ein in der Quanten-Informationstheorie verbreiteter Ansatz gewählt. Nach Einführung eines Modells für Einzel-Qubit-Systeme, der Bloch-Kugel, werden in diesem Kapitel Maße definiert, mit denen sich die Qualität von Messungen quantifizieren lässt: Die Disturbance und die Deviation. Die Disturbance ermöglicht die Angabe der Störung am gemessenen System; die Deviation ermöglicht es, die Güte der Schätzung für den Parameter $|c_1|^2$ eines Qubits anzugeben. Die beiden Maße werden im Verlauf der Diplomarbeit immer wieder verwendet, um Prozeduren zu vergleichen, die das Schätzen von $|c_1|^2$ sowohl zu einem einzelnen Zeitpunkt („Preestimation“ und „Postestimation“) als auch in Echtzeit („Tracking“) ermöglichen. Das Ziel ist üblicherweise das Finden einer optimalen Estimation- und Tracking-Prozedur, d.h. einer Prozedur, bei der zu einer gegebenen Disturbance die Deviation möglichst gering ist.

In Kapitel zwei werden so genannte projektive Messungen zur Verwendung mit Estimation und Tracking diskutiert. Zunächst sieht es so aus, als ob diese aufgrund ihrer stark störenden Natur völlig ungeeignet sind. Insbesondere könnte man denken, dass mit ihnen keine wesentliche Information über die Dynamik eines Qubits gewonnen werden kann. Dem ist jedoch nicht so: Mit projektiven Messungen lässt sich die Frequenz der sinusartigen Oszillation von $|c_1|^2$ mit beliebiger Genauigkeit bestimmen.

Die in Kapitel zwei besprochenen Nachteile projektiver Messungen können, wie in Kapitel drei gezeigt wird, durch Verwendung so genannter minimaler Messungen zum Teil vermieden werden. Im Wesentlichen ist es dadurch möglich, die Disturbance über einen gewissen Bereich kontinuierlich zu regeln. Überraschend und zugleich enttäuschend ist das Ergebnis der zugehörigen optimalen Postestimation-Prozedur: Die beste Schätzung für den Parameter $|c_1|^2$ eines Qubits ist, unabhängig von der verwendeten Disturbance, immer 1/2. Schließlich wird noch kurz ein Thema angesprochen, das von Audretsch et. al. viel diskutiert wurde: Tracking mithilfe von N-Serien. Dabei wird eine Sequenz von kurz aufeinanderfolgenden Messungen zu so genannten N-Serien zusammen gefasst. Deren Ergebnisse werden als Basis für die Schätzung von $|c_1|^2$ in Echtzeit verwendet.

Das letzte große Kapitel der Diplomarbeit ist das vierte (darauf folgen noch die Zusammenfassung mit Ausblick und die Anhänge). Thema sind nun allgemeine Messungen, also Messungen ohne besondere Einschränkungen. Aufgrund ihrer großen Anzahl von Parametern, ist es besonders schwer, optimale Prozeduren für Estimation und Tracking zu finden. Diesbezüglich wird deshalb nur der Fall von Postestimation mit einem und zwei möglichen Messausgängen betrachtet. Es zeigt sich, dass sich mithilfe von unitären Anteilen an den Messoperatoren die Deviation im Vergleich zu jener der entsprechenden Prozedur aus Kapitel drei weiter senken lässt. Die beste Schätzung schwankt dann offenbar zwischen 1/3 und 2/3. Abgeschlossen wird das Kapitel mit der Betrachtung der besten Schätzung für Tracking unter bestimmten Rahmenbedingungen. Sie ermöglicht ein beliebig dichtes Annähern der geratenen Zeitentwicklung an die reale Zeitentwicklung von $|c_1|^2$, auch wenn die verwendeten Messungen nur geringfügig stören.

Contents

Preface

This thesis deals with the estimation and tracking of qubit systems by *sequential quantum measurements* and, related to that, by single quantum measurements. In chapter one the postulates of the theory of quantum mechanics that we are using are presented, followed by discussion of the Bloch sphere, and, finally, the procedures and parameters that are being estimated and tracked. Chapters three, four, and five discuss observation and tracking under certain side conditions that concern the measurement operators. More specifically, chapter three deals with projective measurements, they are fairly easy to handle; In chapter four the complexity is increased due to the use of minimal measurements, a superset of projective measurements; Chapter five deals with the most general but also most complex case, namely general measurements. Finally, there are two appendices. One contains some auxiliary calculations, the other one contains an introduction to the computer program Trasim that was used to create many of the plots in this thesis.

You may wonder what part of this thesis is my original work. I didn't mark it, but note that I usually point to existing results unless I don't know them. Of course, when I say original I don't mean that all the work was based solely on my ideas. Many ideas were already present in the group or evolved in discussions with J. Audretsch, T. Konrad, and also M. Kubitzki. When I started the thesis, I spend some time on the search for good criteria that allow the quantization of the quality of tracking and estimation procedures. Later, I concentrated mostly on several kinds of such procedures and wondered how to optimize these under the criteria that we eventually decided to use (deviation and disturbance). Originally, an idea was to use statistics to reach that goal. Eventually, however, I found that parameterizing procedures and later minimizing them analytically seems to be easier (at least conceptually—the analytic expressions are often quite ugly).

Several calculations in this thesis are only roughly sketched. One reason, I have to admit, is that I was running out of time while writing the thesis (this also serves as an excuse for my bad writing style), another reason is that I think that overly detailed calculations are a bore to the reader. I do, however, have all these (and more) calculations in thorough detail in my personal notes. These notes, although very technical and probably hard to read, are, so I hope, comprehensible for an interested and patient reader. Therefore, I decided to make them available in digital form. To obtain them together with some explanation, a thorough thematic index, and some auxiliary data (mostly calculations done with computer algebra programs), you may try one of the following options:

- Search the World Wide Web (e.g. with `http://www.google.com`) for the term

  `fekleediploma200234`

  and access the corresponding page or pages.
- Contact me by email (as of this writing, my address is `felix.klee@inka.de`).

Note that the computer program *Trasim* is also available through these channels.

Acknowledgments

I want to thank all members of the AG Audretsch for providing a pleasant working atmosphere. In particular, I am indebted to Prof. Dr. J. Audretsch and Dr. Th. Konrad (the official and inofficial supervisors of my thesis) and also to M. Kubitzki (my former roommate). In addition, I want to thank my parents for supporting my studies.

1 Foundations

The purpose of this chapter is the introduction and discussion of fundamental postulates and definitions that are used throughout this thesis.

1.1 The Postulates of Quantum Mechanics

There are several more or less common ways to postulate the theory of Quantum Mechanics. In this thesis we use a way that is similar to that used by M. A. Nielsen and I. L. Chuang in their book "Quantum Computation and Quantum Information" [NC02]. It is based on four postulates, the first of which tells us how to describe physical systems:

State Space Postulate. *Associated with a physical system is a Hilbert space, called* state space. *The state of the system is completely described by a unit vector, called* state vector*, in that space.*

To denote state vectors and related items, we use the *Dirac notation* that is explained in many books on quantum mechanics, for example in the Sakurai [Sak94] and—albeit only for the special case of systems with finite dimensional state spaces—in the Nielsen Chuang [NC02]. In contrast to some books, however, in the present thesis the following convention is used:

> A ket always denotes a *unit* vector. A bra, correspondingly, denotes a unit covector.

You may feel somewhat uncomfortable with the above postulate since it only applies to closed systems and, perhaps, the only "real life" closed system is the universe. Many "real life" systems, however, can be approximated as closed systems, especially when only certain aspects of the systems are of interest. Systems that cannot be approximated as being closed must be treated as subsystems of closed systems. The following two postulates tell us how to construct such systems and how a closed system evolves in time:

Product Space Postulate. *The state space of a system composed of two disjoint subsystems with state spaces $\mathcal{H}_A$, $\mathcal{H}_B$ is the product space $\mathcal{H}_A \otimes \mathcal{H}_B$.*

Time Evolution Postulate. *The* free time evolution *of the state vector $|\psi(t)\rangle$ of a physical system is given by*

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H(t-t_0)} |\psi(t_0)\rangle ,$$

where t_0 is a point in time, $\hbar \approx 1.055\times 10^{-34}\,\mathrm{J\,s}$ [oST04] is the Planck constant divided by 2π, and H, called Hamiltonian, *is a hermitian operator that is characteristic for the system.*

Information about the state of a closed physical system can be extracted with a quantum mechanical measurement explained in the postulate below. Note that we assume without further notice that all measurements in this thesis can be performed in a short period of time during which—aside from disturbance by the measurement—the state of the system barely changes. Therefore, and especially in order to keep things simple, we never take into account the duration of measurements.

Measurement Postulate. *A measurement has a discrete number of outcomes. Associated with each outcome is a* measurement operator *that acts on states in the state space of the system being measured. The only condition that the set of measurement operators $M_0, M_1, \dots, M_{N-1}$ ($N \in \{1, 2, 3, \dots\}$) needs to satisfy in order to describe a valid measurement is the* completeness relation

$$\sum_{m=0}^{N-1} M_m^\dagger M_m = \mathbb{1}.$$

When measuring a system with state vector $|\psi\rangle$, the probability to get outcome m is

$$p(m) = \langle\psi|\, M_m^\dagger M_m \,|\psi\rangle\,.$$

The corresponding postmeasurement state (a measurement may transform the state being measured) is described by the vector

$$|\psi'(m)\rangle = \frac{1}{\sqrt{p(m)}} M_m\,|\psi\rangle\,.$$

1.2 The Bloch Sphere

In the present thesis we deal almost exclusively with systems whose state space is two dimensional. We call these systems *qubits* or, sometimes, *single qubit systems*. It is common to denote the basis vectors of qubit state space by $|0\rangle$ and $|1\rangle$. We use this notation without much further notice throughout this thesis. In this section, however, we will learn how to avoid state space, namely by visualizing states as points on a unit sphere, called Bloch sphere. Before defining that sphere we need to introduce a certain way to parameterize a qubit's state using four real valued parameters:

Lemma 1.2.1. *A qubit state vector $|\psi\rangle$ can be decomposed as follows:*

$$|\psi\rangle = e^{i\chi}\big(\cos\frac{\vartheta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}\,|1\rangle\big), \tag{1.2.1}$$

where

$$\chi \in [0, 2\pi), \varphi \in [0, 2\pi), \vartheta \in [0, \pi].$$

Proof. The basis $\{|0\rangle, |1\rangle\}$ allows us to write

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle, c_j \in \mathbb{C}. \tag{1.2.2}$$

Due to normalization we have

$$\langle\psi|\psi\rangle = 1 \iff |c_0|^2 + |c_1|^2 = 1.$$

It follows that it is possible to define an angle $\theta \in [0, \frac{\pi}{2}]$ such that $|c_0| = \cos\theta$ and $|c_1| = \sin\theta$. By inserting these expressions into equation (1.2.2), we get

$$|\psi\rangle = e^{i\chi} \cos\theta |0\rangle + e^{i\chi'} \sin\theta |1\rangle, \qquad \chi \in [0, 2\pi), \chi' \in [0, 2\pi), \theta \in [0, \frac{\pi}{2}],$$

which, with $\varphi = (\chi' - \chi) \mod 2\pi$ and $\vartheta = 2\theta$, is equivalent to equation (1.2.1). □

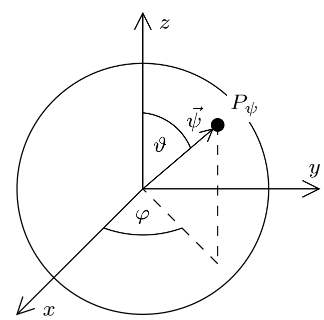

Figure 1.1: Position of the Bloch point corresponding to a state vector $|\psi\rangle = e^{i\chi}\big(\cos\frac{\vartheta}{2}|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|1\rangle\big)$.

Let $|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$ be the state vector of a qubit. Then a decomposition into

$$|\psi\rangle = e^{i\chi}\big(\cos\frac{\vartheta}{2}|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|1\rangle\big), \ \chi \in [0, 2\pi), \varphi \in [0, 2\pi), \vartheta \in [0, \pi],$$

allows us, by interpreting the parameters ϑ and φ as spherical coordinates, to visualize that vector as a point P_ψ on the unit sphere (see figure 1.1). In the context of qubits the unit sphere is called *Bloch sphere*, the vector $\vec{\psi}$ connecting the origin and P_ψ is called *Bloch vector*, and for the point P_ψ itself we use the term *Bloch point*[1].

The mapping from the state vector $|\psi\rangle$ to the Bloch point P_ψ is unique:

- If $c_0 = 0$, then $\vartheta = \pi$ and, consequently, P_ψ coincides with the so called *south pole* of the Bloch sphere (see figure 1.2).
- If $c_1 = 0$, then $\vartheta = 0$ and P_ψ coincides with the *north pole* of the Bloch sphere.
- If $c_0 \neq 0$ and $c_1 \neq 0$, then, because of their limited range, the parameters ϑ and φ are uniquely defined, and so is P_ψ.

The mapping from the Bloch point P_ψ to a state vector, however, is not unique:

- If P_ψ is at the south pole, then $\vartheta = \pi$ and, consequently, $|\psi\rangle = e^{i(\chi-\varphi)}|1\rangle$, where the phase factor $e^{i(\chi+\varphi)}$ is unknown.

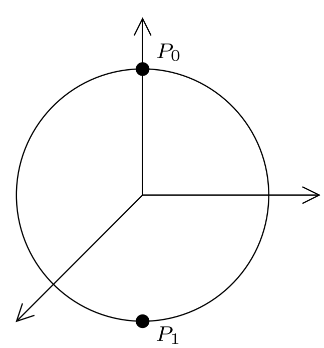

Figure 1.2: The north pole, P_0, and the south pole, P_1, on the Bloch sphere.

[1] Note that in the theory of magnetism the term Bloch point has a completely different meaning: It is used to denote the singularity of a magnetization field [MT02].

- If P_ψ is at the north pole, then $\vartheta = 0$ and $|\psi\rangle = e^{i\chi}|0\rangle$, where $e^{i\chi}$ is unknown.
- If P_ψ is neither at the north nor at the south pole, then the parameters ϑ and φ are well defined and, aside from the factor $e^{i\chi}$, so is $|\psi\rangle = e^{i\chi}\big(\cos\frac{\vartheta}{2}|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}|1\rangle\big)$.

In a nutshell: Given a Bloch point, we can, without further information, infer the corresponding state vector only up to a phase factor. How important is that phase factor? Before answering that question let us look at another lemma:

Lemma 1.2.2. *Two states of that system, $\psi(t_0)$ and $\chi(t_0)$ (t_0 is some point in time), cannot be distinguished by measurement if the corresponding state vectors differ only by a phase factor (i.e. $|\chi(t_0)\rangle = e^{i\xi}|\psi(t_0)\rangle$, where $\varphi \in \mathbb{R}$).*

Proof. To prove that $\psi(t_0)$ and $\chi(t_0)$ are indeed indistinguishable we need to show that the probabilities of a single measurement or a sequence of measurements (which may be started before, at, or after t_0) are the same in two cases. Instead of going into the details of such a proof let us just have a look at what free time evolution and measurement does to two states whose state vectors $|\tilde{\psi}\rangle$ and $|\tilde{\chi}\rangle = e^{i\eta}|\tilde{\psi}\rangle$ differ only by the phase factor $e^{i\eta}$ (it is intention not to discuss only the state vectors $|\psi(t_0)\rangle$ and $|\chi(t_0)\rangle$ because we are not solely interested in what happens at the time t_0). This should give us an idea that the conjecture is true.

- The probability to get a measurement result m (corresponding measurement operator: M_m) when measuring $|\tilde{\chi}\rangle$ is the same as when measuring $|\tilde{\psi}\rangle$:
$$p_{\tilde{\chi}}(m) = \langle\tilde{\chi}| M_m^\dagger M_m |\tilde{\chi}\rangle = \langle\tilde{\psi}| e^{-i\eta} M_m^\dagger M_m e^{i\eta} |\tilde{\psi}\rangle = p_{\tilde{\psi}}(m).$$
- The postmeasurement states $|\tilde{\chi}'_m\rangle$ and $|\tilde{\psi}'_m\rangle$ of a measurement with result m of $|\tilde{\chi}\rangle$ and $|\tilde{\psi}\rangle$, respectively, (corresponding measurement operator: M_m) differ only by the phase factor $e^{i\eta}$:
$$|\tilde{\chi}'_m\rangle = (1/\sqrt{p_{\tilde{\chi}}(m)})M_m|\tilde{\chi}\rangle = (1/\sqrt{p_{\tilde{\psi}}(m)})M_m e^{i\eta}|\tilde{\psi}\rangle = e^{i\eta}|\tilde{\psi}'_m\rangle\,.$$
- The states $|\tilde{\chi}'\rangle$ and $|\tilde{\psi}'\rangle$ resulting from a free time evolution of duration Δt (Hamiltonian H) of $|\tilde{\chi}\rangle$ and $|\tilde{\psi}\rangle$, respectively, differ only by the phase factor $e^{i\eta}$:
$$|\tilde{\chi}'\rangle = e^{-iH\Delta t/\hbar}|\tilde{\chi}\rangle = e^{-iH\Delta t/\hbar}e^{i\eta}|\tilde{\psi}\rangle = e^{i\eta}|\tilde{\psi}'\rangle\,.$$

□

We are now able to answer the question about the importance of the phase factor that made us introduce the above lemma: Since it cannot be measured, the phase factor is not important for the discussion of *observable* features of a qubit's process. In other words:

If we are only interested in the part of a qubit's process that can be detected by measurement then the description of its state by a Bloch point is sufficient. Since this is in general the case, we can always use the "Bloch sphere picture" instead of the "state space picture". What follows is an visualization of the two basic components that any process of a qubit is comprised of: time evolution according to the Time Evolution Postulate , and measurements according to the Measurement Postulate.

1.2.1 Free Time Evolution on the Bloch Sphere

To get a better understanding the dynamics of a qubit that is not subjected to measurements, it pays off to visualize its time evolution on the Bloch sphere. What follows are three lemmas. The first two are necessary for the prove of the third one which states that the trajectory of a freely evolving qubit on the Bloch sphere is very simple to visualize, namely as a circle.

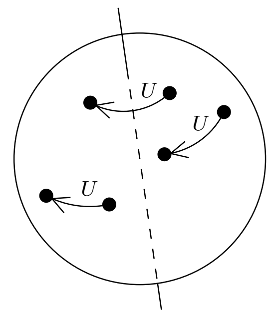

Figure 1.3: On the Bloch sphere, the angle and axis that characterize the rotation by the unitary operator U don't depend on initial state vectors.

Lemma 1.2.3. *If $|\psi\rangle$ is the state vector of a qubit and U is a unitary operator then, on the Bloch sphere, the application of U to $|\psi\rangle$ equals a rotation by an angle $\alpha(U)$ about an axis with a direction $\vec{n}(U)$ that goes through the origin. The quantities $\alpha(U)$ and $\vec{n}(U)$ only depend on U, not on $|\psi\rangle$ (see figure 1.3).*

Proof. See the section "Single Qubit Operations" in the chapter "Quantum Circuits" in the Nielsen Chuang [NC02]. □

Lemma 1.2.4. *The mapping of state vectors $|\psi\rangle$ to Bloch points P_ψ is continuous, in other words*

$$\lim_{|\psi'\rangle \to |\psi\rangle} P_{\psi'} = P_\psi.$$

Proof. By decomposing $P_{\psi'}$ and P_ψ according to lemma 1.2.1 and analyzing the convergence behavior of the cosine, sine, and exponential function terms, for different special case (Bloch point is on north pole, on south pole, neither on south nor on north pole) it follows after some transformations that, in Cartesian coordinates,

$$\lim_{|\psi'\rangle \to |\psi\rangle} \big(\sin\vartheta' \cos\varphi', \sin\vartheta' \sin\varphi', \cos\vartheta'\big) = \big(\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta\big),$$

where ϑ, φ and ϑ', φ' are the spherical coordinates of P_ψ and $P_{\psi'}$, respectively. □

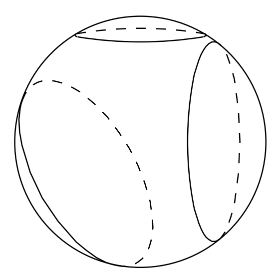

Figure 1.4: Trajectories of qubits on the Bloch sphere.

Lemma 1.2.5. *If $|\psi(t)\rangle$ is the state vector of a qubit that is not subjected to measurements, then its trajectory, when viewed on the Bloch sphere, is equal to a circle (see figure 1.4). The axis of the circle is independent of the state of the qubit at any point in time, i.e. it only depends on its Hamiltonian.*

Proof. Let $P_\psi(t)$ be the point on the Bloch sphere that corresponds to $|\psi(t)\rangle$. To begin with, we observe its trajectory at discrete points in time: $n\,\Delta t$, where n is an integer and Δt is a time interval. According to the Time Evolution Postulate the qubit's evolution between two adjacent points in time, $n\Delta t$ and $(n+1)\Delta t$, is described by the application of the operator $U(\Delta t) = e^{-iH\Delta t/\hbar}$ to $|\psi(n\Delta t)\rangle$. Since $U(\Delta t)$ is a unitary operator we can apply lemma 1.2.3 and we see that $P_\psi(n\Delta t)$ and $P_\psi((n+1)\Delta t)$ are related via a rotation about an axis defined by a vector $n(\vec{U})$. It follows that all points $P(n\Delta t)$ lie on a circle whose axis is independent of the qubits state at any point in time. In the limit $\Delta t \to 0$ the points as a whole form a circle because the time evolution of $|\psi(t)\rangle$ is continuous (by looking at the Time Evolution Postulate it is easily seen that $\lim_{\Delta t\to 0}|\psi(t+\Delta t)\rangle = |\psi(t)\rangle$) and the mapping from state vectors to points on the Bloch sphere is continuous (recall lemma 1.2.4). □

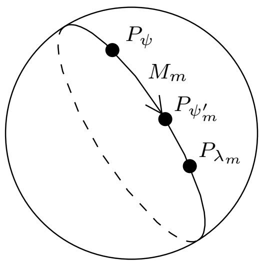

Figure 1.5: A minimal measurement of a qubit in state ψ with result m causes the Bloch point P_ψ to move towards P_{λ_m} ($|\lambda_m\rangle$ is an eigenvector corresponding to the largest eigenvalue of M_m) along a great circle on the Bloch sphere.

1.2.2 Measurements on the Bloch Sphere

Besides time evolution according to the Time Evolution Postulate the other kind of dynamics that a qubit might be subjected to are measurements. To understand how these can be visualized let us first introduce a certain kind of measurement:

> A *minimal measurement* is a measurement where all measurement operators are positive.

Such a measurement can be visualized as follows.

Lemma 1.2.6. *A minimal measurement of a qubit (result: m) causes its Bloch point to move along a great circle towards the Bloch point P_{λ_m} where $|\lambda_m\rangle$ is an eigenvector corresponding to the largest eigenvalue of the measurement operator M_m (see figure 1.5).*

Proof. See the dissertation of T. Konrad [Kon03]. □

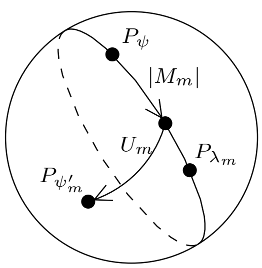

Figure 1.6: By a measurement with result m the Bloch point $P(\psi)$ is transformed according to lemma 1.2.7. Here, $M_m = U_m\,|M_m|$ is a polar decomposition of the corresponding measurement operator, and $|\lambda_m\rangle$ is an eigenvector that corresponds to the largest eigenvalue of $|M_m|$.

The visualization of arbitrary measurements is slightly more complicated:

Lemma 1.2.7. *The transformation of a qubit's Bloch point $P(\psi)$ due to a measurement with result m can, by using a left polar decomposition $U_m\,|M_m|$ of the corresponding measurement operator M_m, be visualized as the result of two successive steps (see figure 1.6):*

1. *The point $P(\psi)$ moves along a great circle towards the point $P(\lambda_m)$, where $|\lambda_m\rangle$ is an eigenvector corresponding to the largest eigenvalue of $|M_m|$.*
2. *The resulting point is rotated by the unitary operator U_m as explained by lemma 1.2.3.*

Proof. The state vector of the qubit after the measurement is

$$|\psi'_m\rangle = \frac{M_m}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}\,|\psi\rangle = U_m \frac{|M_m|}{\sqrt{\langle\psi||M_m|^\dagger|M_m||\psi\rangle}}\,|\psi\rangle\,.$$

We see that the state vector's transformation equals the transformation caused by a minimal measurement with result m and corresponding measurement operator $|M_m|$ followed by the application of the unitary operator U_m. The lemmas 1.2.6 and 1.2.3 tell us how this sequence can be visualized on the Bloch sphere. □

Note that the two steps explained in the above lemma are just a visualization. Recall that the Measurement Postulate does not describe the details of the transformation process.

1.3 Coefficients of a Qubit's State Vector

It has already been said that in this thesis we are mostly dealing with qubits. Often, however, our main focus will not be the state vector $|\psi\rangle = c_0\,|0\rangle + c_1\,|1\rangle$ of a qubit, but the absolute square of the coefficient c_1. To this coefficient we will commonly refer to as *the parameter* $|c_1|^2$ of a qubit. Note that it doesn't make much of a difference whether we deal with $|c_1|^2$ or with $|c_0|^2$ because $|\psi\rangle$ is a unit vector and, therefore, we have the identity

$$|c_0|^2 + |c_1|^2 = 1$$

which allows us to calculate one quantity from the other. With the help of the Bloch sphere we are able to get a somewhat intuitive idea for the relation between a qubit's state vector and the coefficients we just discussed:

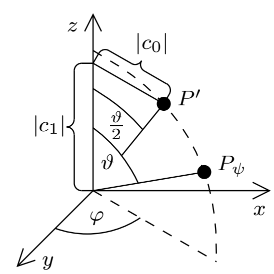

Figure 1.7: Visualization of the parameters $|c_0|$ and $|c_1|$ of a qubit's state vector $c_0\,|0\rangle + c_1\,|1\rangle$ with the help of the Bloch sphere.

Lemma 1.3.1. *In three dimensional space, the absolute values of the coefficients c_0 and c_1 of a qubit's state vector $|\psi\rangle = c_0\,|0\rangle + c_1\,|1\rangle$ can be visualized as follows:*
- *$|c_0|$ is the shortest distance between P' and the z-axis,*
- *$|c_1|$ is the shortest distance between P' and the xy-plane,*

where P' is an auxiliary Bloch point whose azimuthal coordinate is half that of P_ψ and whose polar coordinate is identical to that of P_ψ (see figure 1.7).

Proof. Using the decomposition $|\psi\rangle = e^{i\chi}\big(\cos\frac{\vartheta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}\,|1\rangle\big)$ (see lemma 1.2.1) and the obvious relation $c_j = \langle j|\psi\rangle$ ($j \in \{0,1\}$), we get

$$|c_0| = \left|\cos\tfrac{\vartheta}{2}\right|,\ |c_1| = \left|\sin\tfrac{\vartheta}{2}\right|.$$

That these parameters can be visualized with the help of the Bloch vector P' can be understood by looking at the segment of the unit circle parameterized by ϑ which is displayed in figure 1.7. □

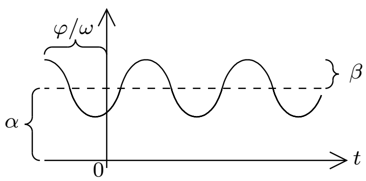

Figure 1.8: Time evolution of $|c_1|^2$ as explained in lemma 1.3.2.

The combination of the above lemma and lemma 1.2.5 ("trajectories of Bloch points are circles") suggests that the time evolution of $|c_1|^2$ is sine-like if the corresponding qubit is not subjected to measurements. This conjecture is quantified in the following lemma:

Lemma 1.3.2. *If $|\psi(t)\rangle = c_0(t)\,|0\rangle + c_1(t)\,|1\rangle$ is the state vector of a qubit with Hamiltonian H that is not subjected to measurements, then (see figure 1.8)*

$$|c_1(t)|^2 = \alpha + \beta\cos(\varphi + \omega t), \tag{1.3.1}$$

where[2]*, using the spectral decomposition*

$$H = E_0\,|E_0\rangle\,\langle E_0| + E_1\,|E_1\rangle\,\langle E_1| \;\; \textit{with} \; \langle E_0|E_1\rangle = 0,\, E_1 \geq E_0,$$

the quantities α, β, φ, and ω are defined as

$$\begin{aligned}
\alpha &= 1 + 2\,|\langle 1|E_0\rangle\,\langle E_0|\psi(0)\rangle|^2 - |\langle 1|E_0\rangle|^2 - |\langle E_0|\psi(0)\rangle|^2\,,\\
\beta &= 2\left|\langle 1|E_0\rangle\,\langle E_0|\psi(0)\rangle\,\langle\psi(0)|1\rangle - |\langle 1|E_0\rangle\,\langle E_0|\psi(0)\rangle|^2\right|,\\
\varphi &= \arg\bigl(\langle 1|E_0\rangle\,\langle E_0|\psi(0)\rangle\,\langle\psi(0)|1\rangle\bigr),\\
\omega &= (E_1 - E_0)/\hbar.
\end{aligned}$$

Proof. Using the Time Evolution Postulate and the identity $c_1(t) = \langle 1|\psi(t)\rangle$, we see that

$$|c_1(t)|^2 = \left|\langle 1|\,e^{-iHt/\hbar}\,|\psi(0)\rangle\right|^2.$$

After inserting the Hamiltonian's spectral decomposition, we get

$$|c_1(t)|^2 = \left|e^{iE_0t/\hbar}\,\langle 1|E_0\rangle\,\langle E_0|\psi_0\rangle + e^{iE_1t/\hbar}\,\langle 1|E_1\rangle\,\langle E_1|\psi_0\rangle\right|^2,$$

which, by application of relations such as $|E_0\rangle\,\langle E_0| + |E_1\rangle\,\langle E_!| = \mathbb{1}$, can be shown to be equal to equation (1.3.1). □

1.4 Estimation and Tracking

Now, we are ready to introduce one of the main topics of this thesis: The estimation and tracking of qubits. What is meant by that? We define *estimation* to be the theory of finding out what parameter $|c_1|^2$ the qubit has at a certain point in time. *Tracking* is the theory of tracking $|c_1|^2$ continuously in *real time*. As you see, estimation and tracking are *not* concerned with finding the entire state vector of a qubit.

[2] We use the common definition $\arg z = \varphi \in [0, 2\pi) : |z|\,e^{i\varphi} = z$.

1.4.1 Estimation

In the chapters to come we will see many different procedures for the estimation of a qubit. Essentially, they all consist of two steps:

1. The qubit is measured using a single measurement or a sequence of measurements.
2. Using the measurement results and, maybe, further information (for example it may be known that the qubits initial state was in the northern hemisphere of the Bloch sphere) the qubit's parameter $|c_1|^2$ at a certain point in time is estimated by a guess, commonly denoted as g_1.

Basically, we distinguish two kinds of estimation procedures:

Preestimation procedures: These are procedures where $|c_1|^2$'s value as it was right before the measurement(s) is guessed.

Postestimation procedures: These are procedures where $|c_1|^2$'s value as it was right after the measurement(s) is guessed.

In order to compare procedures of the same kind we need a good measure for quality. It is, however, hard to find a single real number for that purpose. The reason is that we are interested in two aspects of an estimation procedure: How much is a qubit's state disturbed during measurement, and how much does the guess g_1 deviate from $|c_1|^2$? That's why we introduce the following parameters (the mean value is explained below):

Estimation disturbance: Mean square deviation between $|c_1'|^2$ and $|\tilde{c}_1'|^2$, commonly denoted[3] as $s = \overline{(|c_1'|^2 - |\tilde{c}_1'|^2)^2}$. The parameter $|c_1'|^2$ is $|c_1|^2$ after the measurement process, and $|\tilde{c}_1'|^2$ is what $|c_1|^2$ would look after measurement process if it hadn't been disturbed by any measurements.

Preestimation deviation: Mean square deviation between g_1 and $|c_1|^2$, commonly denoted as $v = \overline{(g_1 - |c_1|^2)^2}$. The value of g_1 is, of course, the guess for $|c_1|^2$ as it was before the measurement process.

Postestimation deviation: Mean square deviation between $g_1{\prime}$ and $|c_1'|^2$, commonly denoted as $v' = \overline{(g_1' - |c_1'|^2)^2}$. The value of g_1' is the guess for $|c_1'|^2$, where, as above, $|c_1'|^2$ is the value of $|c_1|^2$ after the measurement procedure.

The mean value in all these cases, is to be taken over

- all possible initial states,
- all possible measurement outcomes,

[3]You may wonder why I chose the letters s and v to denote these quantities. The reason is that the words "disturbance" and "deviation" share their first and second letters, so I chose their third letters to distinguish them.

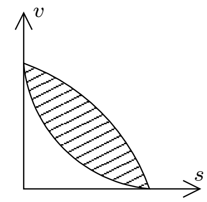

Figure 1.9:
Disturbance-deviation-graph of a (fictive) parameterized estimation procedure. Each point in the shaded area corresponds to one or more parameter combinations.

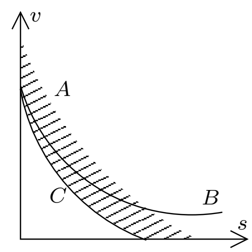

Figure 1.10:
Disturbance-deviation-graphs. The procedures A and C are better than B. C is better than A and may be optimal.

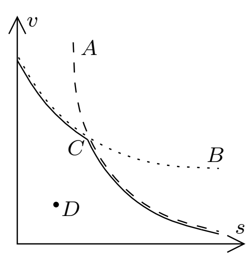

Figure 1.11: Disturbance-deviation graphs. The procedure A is not better than B, nor is B better than A, but C is better than A and B, and D is better than C. D may be optimal.

- all possible time evolutions of the system in between measurements (only necessary for sequences of measurements).

Related to that mean value, are the following expressions that we often use when characterizing a qubit:

- A qubit with a *totally unknown state* is one, where the probability density, often denoted as $\varrho(\psi)$, to find its state is everywhere the same on the Bloch sphere. The function $\varrho(\psi)$ is, in this case, frequently called *uniform probability density*.
- A qubit is said to have *totally unknown time evolution* if the probability density $\varrho(U)$ (U denotes a time evolution operator $e^{-iHt/\hbar}$) is *uniform* which, by definition, is the case when $\varrho(U\,|0\rangle)$ is uniform (i.e. the probability density to find a state with state vector $U\,|0\rangle$ is everywhere the same on the Bloch sphere).

It must be said that the procedures that we discuss in this thesis cannot be assigned only one disturbance-deviation-pair. The reason is that they all are parameterized (e.g. all procedures provide parameters that allow the control of the measurements used). So, it is best to think of a *disturbance-deviation-graph* assigned to a procedure. In figure 1.9 a possible such graph is displayed.

The way how to compare the quality follows naturally:

> An estimation procedure A is defined to be *better* than a procedure B if, for each possible disturbance s that A and B can adopt, the relation
>
> $$v^{(A)}_{\max}(s) \le v^{(B)}_{\min}(s)$$
>
> is satisfied, and, at least for one disturbance s, the relation
>
> $$v^{(A)}_{\max}(s) < v^{(B)}_{\min}(s)$$
>
> is satisfied ($v^{(A)}_{\max}(s)$ and $v^{(B)}_{\min}(s)$ are the largest and smallest corresponding deviations, respectively).

Accordingly, we define an estimation procedure to be *optimal*, if there is no better such procedure. Figures 1.10 and 1.11 provide examples for the comparison of estimation procedures. The pathologic cases being illustrated by the graphs don't need to worry us.

Sometimes it proves to be easier to find a procedure that in addition to being optimal also satisfies the following condition

> There is no other estimation procedure that for any possible deviation has a lower corresponding disturbance.

Such a procedure is called *superoptimal* (see figure 1.12 for an example).

Some final note concerning the terms optimality and superoptimality. We will not always attribute these plainly to estimation procedures. For example, we will frequently use the term "*optimal parameters*" by which we mean the set of parameters that make an estimation procedures as good as it can be (see figure 1.13). Sometimes used is the term "*best*" in conjunction with estimation procedures or their parameters. In this case, it just is a synonym for the term "optimal". The term "optimal guess" refers to the guess in the set of optimal parameters.

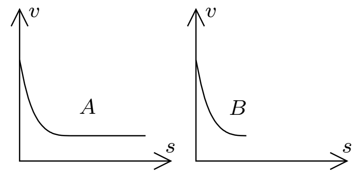

Figure 1.12: If the estimation procedure A is optimal, then B is superoptimal.

1.4.2 Tracking

A procedure for tracking a qubit consists, in general, of the repetition of the following steps:
- The qubit is measured.
- During the time up to the next measurement or up to the end of the procedure, the qubit's parameter $|c_1|^2$ is estimated by a guess, commonly denoted as g_1, that may take into account all previous measurement results and any other information about the qubit.

See figure 1.14 for a fictive example of the devolution of a tracking procedure.

To characterize the quality of tracking procedures, we introduce parameters similar to those used for estimation procedures:

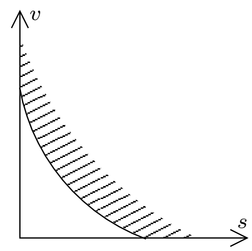

Figure 1.13: The optimal parameters of the procedure with the displayed disturbance deviation graph are those that "produce" the lowest part (marked by a solid curve).

Tracking Disturbance: Mean square deviation between $|c_1(t)|^2$ and $|\tilde{c}_1(T)|^2$. $|c_1(t)|^2$ is the value of $|c_1|^2$ at the time t, and $|\tilde{c}_1(t)|^2$ is what $|c_1(t)|^2$ would look if $|c_1|^2$ hadn't been subjected to any measurements by the tracking procedure.

Tracking deviation: Mean square deviation between the guess $g_1(t)$ and $|c_1(t)|^2$ ($|c_1(t)|^2$ is the same as above).

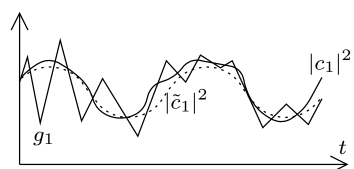

Figure 1.14: Evolution of the undisturbed state $|\tilde{c}_1|^2$, the actual state $|c_1|^2$, and the guess g_1 in a tracking procedure.

The mean value has to be taken over
- the duration of the measurement,
- all possible initial state,
- all possible sequences of measurement outcomes,
- all possible time evolutions of the system in between measurements.

We say that a tracking procedure A is better than a tracking procedure B if, for each possible disturbance s, the corresponding largest and smallest deviations satisfy the relation $v_{\max}^{(A)}(s) \leq v_{\min}^{(B)}(s)$ and, at least for one disturbance s, $v_{\max}^{(A)}(s) < v_{\min}^{(B)}(s)$. The *optimality of a tracking procedure* is defined accordingly.

It has to be said that in this thesis, we never calculate nor discuss the disturbance and deviation of a tracking procedure. The reason is that, due to the required time average, the resulting expressions can become very complicated. In addition, we can approximate a tracking procedure as a set of overlapping postestimation procedures:

> The first procedure is based on the first measurement, the second procedure on the first two measurements, the third on the first three, and so on.

This is actually the reason why there is so much discussion of postestimation procedures in this thesis. When I started research, the main objective was to find good tracking procedures. Incidentally, since I decided that preestimation procedures are not of much use for the discussion of tracking, they aren't discussed in this thesis. Anyone that wants to learn more about them is referred to the diploma thesis of M. Kubitzki [Kub03].

2 Estimation and Tracking with Projective Measurements

The present chapter could be seen as a kind of warm-up for later chapters where—due to generality—things sometimes get very complicated. Instead of discussing a broad range of different measurements, we will limit ourselves to so called projective measurements, a class of measurements that you might already be familiar with and that, usually, is fairly easy to handle:

> A *projective measurement* is a measurement whose measurement operators $M_0, M_1, \ldots, M_{N-1}$ are of the form
>
> $$M_m = |\lambda_m\rangle\langle\lambda_m| \text{ with } \langle\lambda_m|\lambda_{m'}\rangle = \delta_{mm'}.$$

The name of the measurement goes back to the fact that, since the measurement operators are projectors, they *project* the premeasurement state into a postmeasurement state that is—aside from a phase factor—independent of the premeasurement state:

Lemma 2.0.1. *The postmeasurement state vector of a qubit that has been subjected to a projective measurement (measurement operators: $M_j = |\lambda_j\rangle\langle\lambda_j|$, $j \in \{0,1\}$) is*

$$|\psi'\rangle = \mathrm{sign}(\langle\lambda_m|\psi\rangle)\,|\lambda_m\rangle\,, \tag{2.0.1}$$

where m is the readout of the measurement, and ψ is the premeasurement state of the qubit.

Proof. Equation (2.0.1) follows by the application of the Measurement Postulate. □

An obvious consequence of this lemma is that, knowing the result of a measurement and its measurement operators, we can tell the measurement state (up to a phase factor) with certainty. We don't need to know the premeasurement state!

Something else that is peculiar about projective measurements is the fact that the number of measurement operators $M_0 = |\lambda_0\rangle\langle\lambda_0|$, $M_1 = |\lambda_0\rangle\langle\lambda_0|$,... is related to the dimension of the Hilbert space that they operate on. In single qubit Hilbert space their number is exactly two. As is easily seen, this follows from the condition $\langle\lambda_m|\lambda_{m'}\rangle = \delta_{mm'}$ and the completeness relation $\sum_m M_m^\dagger M_m = \sum_m |\lambda_m\rangle\langle\lambda_m| = \mathbb{1}$.

Figure 2.1 shows a visualization of the projective measurement on the Bloch sphere. Note that $|\lambda_0\rangle$ and $|\lambda_1\rangle$ are diametrically opposing points on the Bloch sphere. This is not by accident:

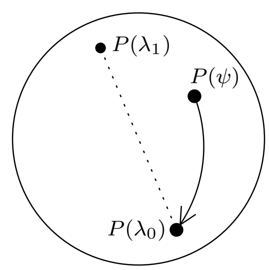

Figure 2.1: A projective measurement (measurement operators: $M_0 = |\lambda_0\rangle\langle\lambda_0|, M_1 = |\lambda_1\rangle\langle\lambda_1|$)) with result 0 projects a state vector $|\psi\rangle$ onto $|\psi'\rangle = |\lambda_0\rangle$.

Lemma 2.0.2. *Any two orthogonal single qubit states $|\psi_0\rangle$, $|\psi_1\rangle$ are diametrically opposing points P_0, P_1 on the Bloch sphere.*

Proof. According to lemma 1.2.1 we can write the states $|\psi_0\rangle, |\psi_1\rangle$ as

$$|\psi_j\rangle = e^{i\chi_j}(\cos\frac{\vartheta_j}{2}|0\rangle + e^{i\varphi_j}\sin\frac{\vartheta_j}{2}|1\rangle),\ j \in \{0,1\},$$

where $\vartheta_j \in [0,\pi]$, $\varphi_j \in [0,2\pi]$ constitute their spherical coordinates on the Bloch sphere. Since the states are orthogonal, it follows that $\langle\psi_0|\psi_1\rangle = 0$ and, using the above expressions, we get

$$\cos\frac{\vartheta_0}{2}\cos\frac{\vartheta_1}{2} + e^{i(\varphi_1-\varphi_0)}\sqrt{1-\cos^2\frac{\vartheta_0}{2}}\sqrt{1-\cos^2\frac{\vartheta_1}{2}} = 0.$$

The solution of this relation in terms of ϑ_j, φ_j is

$$(\vartheta_0 = \pi \vee \vartheta_0 = 0 \vee \varphi_1 = \varphi_0 - \pi \vee \varphi_1 = \varphi_0 + \pi)\ \wedge\ \vartheta_1 = \pi - \vartheta_0.$$

After calculating the Bloch points P_0, P_1 (in Cartesian coordinates: $P_j = (\sin\vartheta_j\cos\varphi_j, \sin\vartheta_j\sin\varphi_j, \cos\vartheta_j)$) for each of the four possible solutions, it follows that $P_1 = -P_0$. □

We will now apply out knowledge about projective measurements to estimation and tracking. In the following two sections we will discuss an estimation and a tracking procedure under the following side conditions:

- The initial state of observed systems is completely unknown.
- The time evolution of observed systems is completely unknown.

For each procedure we will find the optimal parameters. Finally, we will turn to a completely different kind of procedure, which we call frequency estimation procedure.

2.1 Postestimation

One of the simplest postestimation procedures is a procedure that employs only a single projective measurement:

SPMP Procedure.
Full name: Single Projective Measurement Postestimation Procedure.
Parameters:

- Projective measurement operators $M_j = |\lambda_j\rangle\langle\lambda_j|, j \in \{0,1\}$.
- A guess function g_1.

Steps:

1. A projective measurement (measurement operators:) is performed on a single qubit system in a premeasurement state $|\psi\rangle$. We denote the readout by m and the postmeasurement state by $|\psi'_m\rangle$.
2. $|c'_1(m)|^2 = |\langle 1|\psi'_m\rangle|^2$ is estimated by a guess $g'_1(m) \in \mathbb{R}$ that may depend on

- the measurement outcome m,
- the measurement operators M_0, M_1,
- information, if available, about the premeasurement state $|\psi\rangle$.

Disturbance-deviation graphs for SPMP procedures with different parameters are displayed in figure 2.2. In order to draw such graphs and, ultimately, to find an optimal SPMP procedure, we need to know the disturbance and postestimation deviation:

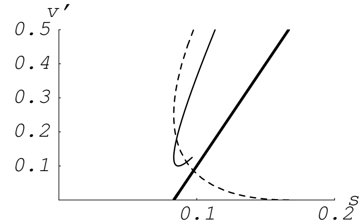

Figure 2.2: SPMP procedures with $g_1' = \alpha m, \alpha \in [0.3, 0.8]$ (solid thin curve), $g_1' = m, \alpha \in [0, 1]$ (dashed curve), and $g_1' = \alpha, \alpha \in [0, 0.6]$ (thick curve). The parameter α corresponds to the measurement operator $M_0 = |\lambda_0\rangle\langle\lambda_0|$: $\alpha = |\langle 1|\lambda_0\rangle|^2$.

Lemma 2.1.1. *When applied to a single qubit system whose initial state is completely unknown an SPMP procedure has the following disturbance s and postestimation deviation v' (note that we use the same notation as in the definition of the SPMP procedure):*

$$\begin{aligned} s =& \frac{1}{6}(2\alpha^2 - 2\alpha + 1), \qquad (2.1.1)\\ v' =& \frac{1}{2}\big(g_1'(m=0)^2 + g_1'(1)^2 - 2g_1'(1) + 1\big)\\ &- \big(g_1'(0) - g_1'(1) + 1\big)\alpha + \alpha^2, \qquad (2.1.2) \end{aligned}$$

where

$$\alpha = |\langle 1|\lambda_0\rangle|^2 . \qquad (2.1.3)$$

Proof. In the last chapter we have defined the disturbance and postestimation deviation to be the average of $(|c_1'(m,\varphi)|^2 - |c_1(\psi)|^2)^2$ and $(g_1'(m) - |c_1'(\psi)|^2)^2$, respectively. In the present case we need to average over the premeasurement states and the two possible measurement results. Therefore we have

$$s = \int d\psi\, \varrho(\psi) \sum_m p(m|\psi)\big(|c_1'(m,\psi)|^2 - |c_1(\psi)|^2\big)^2, \qquad (2.1.4)$$

$$v' = \int d\psi\, \varrho(\psi) \sum_m p(m|\psi)\big(g_1'(m) - |c_1'(m,\psi)|^2\big)^2, \qquad (2.1.5)$$

where $\varrho(\psi)$ is the probability density of the premeasurement states and $p(m|\psi)$ is the probability to get measurement result m, if the premeasurement state is $|\psi\rangle$.

Let us now prove that expression 2.1.4 is equal to expression 2.1.1 (the prove that 2.1.5 is equal to 2.1.2 is very similar—so we omit it). First, we rewrite 2.1.4 as a sum:

$$\begin{aligned} s &= s_1 - 2s_2 + s_3, \qquad (2.1.6)\\ s_1 &= \textstyle\int d\psi\, \varrho(\psi) \sum_m p(m|\psi)\, |c_1'(m,\psi)|^2 ,\\ s_2 &= \textstyle\int d\psi\, \varrho(\psi) \sum_m p(m|\psi)\, |c_1'(m,\psi)|^2\, |c_1(\psi)|^2 ,\\ s_3 &= \textstyle\int d\psi\, \varrho(\psi) \sum_m p(m|\psi)\, |c_1(\psi)|^4 . \end{aligned}$$

The basic building blocks of the terms s_1, s_2, s_3 are

$$
\begin{aligned}
&p(m|\psi) = \langle\psi| M_m^\dagger M_m |\psi\rangle = |\langle\lambda_m|\psi\rangle|^2\,,\\
&|c_1(\psi)|^2 = |\langle 1|\psi\rangle|^2\,,\\
&|\psi'(m,\psi)\rangle = \tfrac{1}{\sqrt{p(m|\psi)}} M_m |\psi\rangle = \tfrac{\langle\lambda_m|\psi\rangle}{\sqrt{p(m|\psi)}} |\lambda_m\rangle\,,\\
&|c_1'(m,\psi)|^2 = |\langle 1|\psi'(m,\psi)\rangle|^2 = |\langle 1|\lambda_m\rangle|^2\,.
\end{aligned}
$$

By substituting them into s_1, s_2, s_3 we get[1]

$$
\begin{aligned}
s_1 &= \textstyle\sum_m |\langle 1|\lambda_m\rangle|^4 \int d\psi\, \varrho(\psi)\, |\langle\lambda_m|\psi\rangle|^2 = \sum_m |\langle 1|\lambda_m\rangle|^4 \int_0^1 d\alpha\, \alpha\\
&= \textstyle\sum_m |\langle 1|\lambda_m\rangle|^4 \, \frac{1}{2},
\end{aligned}
$$

$$
\begin{aligned}
s_2 &= \textstyle\sum_m |\langle 1|\lambda_m\rangle|^2 \int d\psi\, \varrho(\psi) \left|\langle\lambda_m| \big(|\psi\rangle\langle 1|\big) |\psi\rangle\right|^2\\
&= \textstyle\sum_m |\langle 1|\lambda_m\rangle|^2 \left(\frac{1}{6}|\langle 0|\lambda_m\rangle|^2 + \frac{1}{3}|\langle 1|\lambda_m\rangle|^2\right)\\
&= \textstyle\sum_m |\langle 1|\lambda_m\rangle|^2 \left(\frac{1}{6}\langle\lambda_m|\mathbb{1}|\lambda_m\rangle + \frac{1}{6}\langle\lambda_m|1\rangle\langle 1|\lambda_m\rangle\right)\\
&= \textstyle\frac{1}{6}\sum_m \big(|\langle 1|\lambda_m\rangle|^2 + |\langle 1|\lambda_m\rangle|^4\big),
\end{aligned}
$$

$$
\begin{aligned}
s_3 &= \textstyle\int d\psi\, \varrho(\psi) \sum_m |\langle\lambda_m|\psi\rangle|^2 |\langle 1|\psi\rangle|^4 = \int d\varrho(\psi)\, \langle\psi|\mathbb{1}|\psi\rangle\, |\langle 1|\psi\rangle|^4\\
&= \textstyle\int d\psi\, \varrho(\psi)\, |\langle 1|\psi\rangle|^4 = 1/3.
\end{aligned}
$$

With these expressions the sum (2.1.6) becomes

$$
s = \tfrac{1}{6}\big(|\langle 1|\lambda_0\rangle|^4 - 2|\langle 1|\lambda_1\rangle|^2 + |\langle 1|\lambda_1\rangle|^4 - 2|\langle 1|\lambda_1\rangle|^2\big) + \tfrac{1}{3},
$$

and, by using the relations

$$
\begin{aligned}
&|\langle 1|\lambda_0\rangle|^2 \overset{(2.1.3)}{=} \alpha,\\
&|\langle 1|\lambda_1\rangle|^2 = \langle 1|\lambda_1\rangle\langle\lambda_1|1\rangle = \langle 1|\big(\mathbb{1} - |\lambda_0\rangle\langle\lambda_0|\big)|1\rangle = 1-\alpha,
\end{aligned}
$$

it can be simplified to

$$
s = \tfrac{1}{6}(2\alpha^2 - 2\alpha + 1),
$$

which is identical to expression (2.1.1). □

What are the optimal parameters for the SPMP procedure? Let us start with the best guess:

Lemma 2.1.2. *If an SPMP procedure is used to estimate a totally unknown[2] qubit, then its optimal guess is (we use the same notation as in the definition of the SPMP procedure)*

$$
g_1'(m) = |\langle 1|\lambda_m\rangle|^2\,.
$$

[1] Note that some steps in the calculations are a bit rough. Originally, I planned to describe them in appendix 5, but as time was running out, I decided against it.

[2] Actually this restriction is not necessary: The optimal guess derived here is independent of the amount of information that we have about the initial system. However, this generalization is not of importance.

Proof. According to lemma 2.0.1, after the measurement the observed system is in the state $|\psi'_m\rangle$ which is physically identical to $|\lambda_m\rangle$. Consequently, $|c'_1(m)|^2 = |\langle 1|\lambda_m\rangle|^2$. Because this quantity doesn't depend on the system's initial state, it can be guessed with certainty - the optimal guess is obviously

$$g'_1(m) = |\langle 1|\lambda_m\rangle|^2 . \tag{2.1.7}$$

Since $g'_1(m) = |c'_1(m)|^2$, we expect the postestimation deviation to vanish. This is indeed the case as can be seen by inserting (2.1.7) into equation (2.1.2):

$$v'\left(g'_1(m) = |\langle 1|\lambda_m\rangle|^2\right) = 0.$$

□

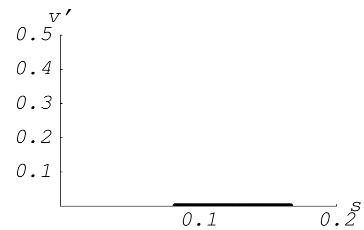

Figure 2.3: Disturbance-deviation graph of the SPMP procedure with optimal parameters.

With the above guess the disturbance-deviation graph of the SPMP procedure looks as displayed in figure 2.3. It follows that no further restrictions to the parameters are necessary. In other words, the following parameters are optimal:

- Any two projective measurement operators, $M_0 = |\lambda_0\rangle\langle\lambda_0|, M_1 = |\lambda_1\rangle\langle\lambda_1|$ that satisfy the completeness relation according to the Measurement Postulate.
- The guess $g'_1(m) = |\langle 1|\lambda_m\rangle|^2$.

2.2 Uniform Tracking

Although is carries a load of disadvantages, the following simple tracking procedure is quite instructive:

UPMT Procedure.
Full name: Uniform Projective Measurement Tracking Procedure.
Parameters:

- A set of projective measurement operators, $\{M_0 = |\lambda_0\rangle\langle\lambda_0|, M_1 = |\lambda_1\rangle\langle\lambda_1|\}$.
- The number of measurements, N.
- The distance in time between measurements, τ.
- A guess function, g_1.

Steps: The state $\psi(t)$ of the qubit under observation is measured at times t_0, $t_0 + 1\tau$, $t_0 + 2\tau$, …, $t_0 + (N-1)\tau$, where t_0 is the time when tracking starts. In parallel, the evolution of the parameter $|c_1(t)|^2 = |\langle 1|\psi(t)\rangle|^2$ is continuously estimated using the guess function $g_1(t)$ which may take into account all previous measurement results and any additional information about the system.

First, a remark concerning the name of the procedure: it is called "uniform" because all measurements use the same measurement operators and each consecutive two measurements are separated by the same time interval. Let us now discuss the evolution of the parameter $|c_1(t)|^2$ of a qubit under the influence of a UPMT procedure. It is illustrated in figure 2.4. The system in all three plots has the same Hamiltonian and the same initial state.

In the top plot of figure 2.4 we see that, with the onset of tracking, the vertical offset, the amplitude, and the phase of $|c_1(t)|^2$ are "destroyed" as a consequence of the projective measurements. Only the frequency is preserved since, recall lemma 1.3.2, it only depends on the qubit's Hamiltonian, not it's state.

In the middle plot of figure 2.4, the same measurement operators as in the top plot are used. However, they are applied at a very high rate. The result is that the qubit's state barely evolves in between measurements. Therefore, the probability to get the same measurement result twice in a row is very high, and, consequently, the parameter $|c_1|^2$ stays the same for long periods of time. The effect that the dynamics of a system can be suppressed by a quick succession of projective measurements is known as *quantum Zeno effect*.

In the bottom plot of figure 2.4 we see that the dynamics of the system is suppressed, although the measurements are applied with the same rate as in the top plot. What is happening here? Recall from lemma 1.2.5 that the time evolution of the qubit's Bloch point can be described as a rotation about an axis whose orientation is defined solely by the Hamiltonian of the system. Now, if, by measurement, the Bloch point is projected onto that axis, it does not show any dynamics anymore. This is what is happening here.

What are the optimal parameters for the UPMT procedure? We will discuss only the optimal guess. In the case of a quick succession of measurements (measurement operators: $M_0 = |\lambda_0\rangle\langle\lambda_0|$, $M_1 = |\lambda_1\rangle\langle\lambda_1|$), it is obviously

$$g_1 = |\langle 1|\lambda_m\rangle|^2 ,$$

where m is the last measurement readout. In the case that the measurements are more widely spaced as for example displayed in the top plot of figure 2.4 this guess does not seem to be appropriate anymore. Then, the guess

$$g_1 = 1/2$$

seems to be the optimal guess since, on average, the distance of $|c_1(t)|^2$ to 1/2 is smaller than to any other value. Which guess should be chosen depends, of course, on a priory information about the minimum expected frequency of $|c_1(t)|^2$. In any case, it does not seem to be possible to get much information about the original dynamics of $|c_1(t)|^2$ by using the UPMT procedure. Additionally, this procedure has the disadvantage that, due to the disruptive character of projective measurements, the disturbance is very high, even for short tracking durations.

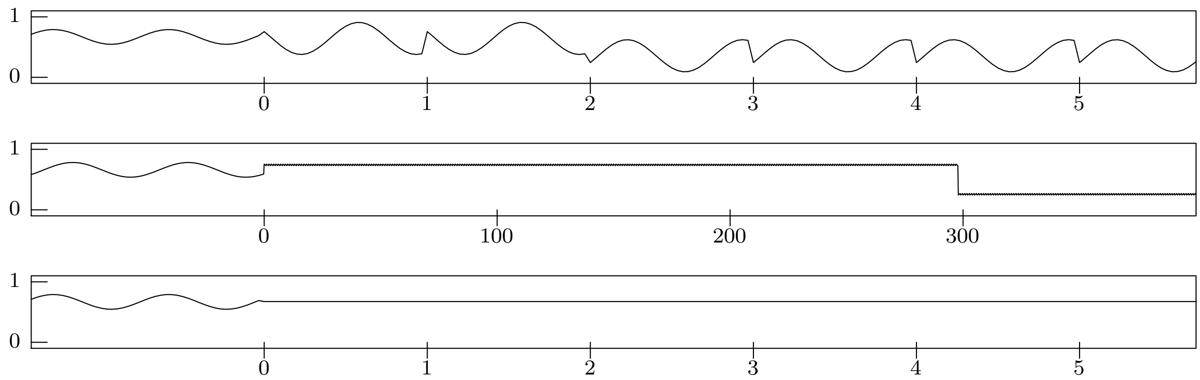

Figure 2.4: Evolution of the parameter $|c_1(t)|^2$ of a qubit under the influence of the UPMT procedure. On the vertical axis, the time axis, individual measurements are denoted by whole numbers.

2.3 Frequency Determination

Judging from the last section, one may come to the conclusion that, using projective measurements, it is not possible to get any valuable information about the time evolution of a qubit's parameter $|c_1|^2$. This is not true! In fact, as we will see below, it is possible to determine the frequency of the oscillation of $|c_1|^2$ to arbitrarily high precision with the following procedure:

FD Procedure.
Full name: Frequency Determination Procedure.
Definitions:

- State of the observed qubit: $|\psi\rangle$.
- Offset α, amplitude β, phase φ, frequency ω, such that (recall lemma 1.3.2)

$$|c_1(t)|^2 = \alpha + \beta \cos(\varphi + \omega t).$$

- Observable frequency[3]:

$$\tilde{\omega} = \begin{cases} 0 & \text{if the system's Hamiltonian rotates Bloch points} \\ & \text{about the } z\text{-axis of the Bloch sphere,} \\ \omega & \text{otherwise.} \end{cases}$$

- Measurement operators: $M_0 = |0\rangle\langle 0|$, $M_1 = |1\rangle\langle 1|$.

Parameters:

- Measure for the time interval between measurements: $\Delta t \in (0, \Delta t_{\max}]$, where $\Delta t_{\max} = 2\pi/\tilde{\omega}$ for $\tilde{\omega} \neq 0$. For $\tilde{\omega} = 0$ the value of $\Delta t_{\max}$ can be chosen at will, it only must be positive.

[3] This definition is used because, if the Hamiltonian rotates states about the Bloch sphere, there is no chance to determine the frequency ω by observing $|c_1|^2$.

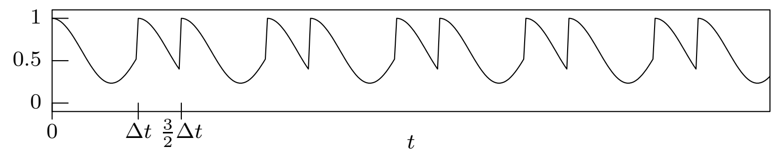

Figure 2.5: The evolution of the parameter $|c_1|^2$ of a qubit under the influence of an FD procedure with alternating durations between measurements (simulated with Trasim).

- Measure for the number of measurements: $N \in \{1, 2, 3, \dots\}$.

Steps:

1. The variables $M(\frac{\Delta t}{2})$, $M(\Delta t)$ are initialized to 0.
2. For preparation, the state $|\psi(t)\rangle$ is measured using the set of measurement operators $\{M_0, M_1\}$. The readout is denoted by m.
3. The following steps are repeated N times for $\tau = \frac{\Delta t}{2}$ and N times for $\tau = \Delta t$ (the order doesn't matter):
   a) In order to transform the vector $|\psi\rangle$ into $|1\rangle$, the unitary operator $U = |1\rangle\langle 0| + |0\rangle\langle 1|$ is applied to $|\psi\rangle$ if $m = 0$ (this can be done by performing a measurement with U as its only measurement operator).
   b) The current time, t, is stored in the variable $\tilde{t}$.
   c) During a time period of duration τ, the qubit's state vector evolves into $|\psi(\tilde{t} + \tau)\rangle$.
   d) $|\psi(\tilde{t} + \tau)\rangle$ is measured using the set of measurement operators $\{M_0, M_1\}$. The readout is denoted by m.
   e) If $m = 1$, then $M(\tau)$ is incremented by 1.
4. The observable frequency $\tilde{\omega}$ is approximated by[4]

$$\tilde{\omega}_{\mathrm{aprx}} = \frac{2}{\Delta t} \arccos f(a_{\mathrm{aprx}}, b_{\mathrm{aprx}}),$$

where $a_{\mathrm{aprx}} = M(\frac{\Delta t}{2})/N$, $b_{\mathrm{aprx}} = M(\Delta t)/N$, and

$$f(x, y) = \begin{cases} \frac{y-1}{2(x-1)} - 1 & \text{for } x \in [0, 1) \wedge y \in [4x - 3, 1], \\ 1 & \text{for } x = 1 \wedge y = 1, \\ 1 & \text{for } x \in [0, 1] \wedge y \in [0, 4x - 3). \end{cases}$$

To get a feeling how the FD procedure affects the evolution of the parameter $|c_1|^2$, let us have a look at figure 2.5. We see that, with the onset of the procedure at $t = 0$, $|c_1|^2$ is projected into 1 (steps 2 and 3a). Then $|c_1|^2$ evolves freely for a time period $\tau = \Delta t$ (step 3c) until it is measured (step 3d) and, at the same time, projected again into 1 (steps 3d and 3a). This process is repeated with alternating values of $\tau \in \{\Delta t, \Delta t/2\}$. Note that, due to symmetry, step 3a

[4] We use the common definition $\arccos : x \to \varphi$ with $\varphi \in [0, \pi]$ and $\cos\varphi = x$.

could be avoided (in this case $|c_1|^2$ would sometimes be projected into 0) but then the calculation of $\tilde{\omega}_{\mathrm{aprx}}$ in step 4 may become a bit more complicated. Something else that could be improved is the first measurement (step 2): Its readout, m, is thrown away. This value, however, could be used to get a little information about the offset α, the amplitude β, and the phase φ. A preceding measurement specifically designed for that purpose could be a further improvement. Let us conclude this discussion by quantifying how the parameter $|c_1|^2$ evolves under the influence of an FD procedure:

Lemma 2.3.1. *The parameter $|c_1|^2$ of a qubit with Hamiltonian H that is subjected to an FD procedure evolves, in between measurements at times $\tilde{t}$, as follows:*

$$\left|c_1(\tilde{t}+\tau)\right|^2 = 1 - \beta' + \beta' \cos(\omega\tau), \tag{2.3.1}$$

where the value $\beta' = 2\big(|\langle 1|E_0\rangle|^2 - |\langle 1|E_0\rangle|^4\big)$ is based on the spectral decomposition

$$H = E_0 \left|E_0\right\rangle \left\langle E_0\right| + E_1 \left|E_1\right\rangle \left\langle E_1\right| \ \text{ with } \langle E_0|E_1\rangle = 0,\ E_1 \geq E_0.$$

Proof. According to lemma 1.3.2 the time evolution of $\left|c_1(\tilde{t}+\tau)\right|^2$ is given by

$$\left|c_1(\tilde{t}+\tau)\right|^2 = \alpha' + \beta' \cos(\varphi' + \omega\tau),$$

where α', β', φ', and ω are defined by the eigensystem E_j, $|E_j\rangle$ ($j \in \{0,1\}$) and by the initial state $|\psi(\tilde{t})\rangle$. With $|\psi(\tilde{t})\rangle = |1\rangle$ if follows, after basic algebraic manipulations, that equation (2.3.1) is true. □

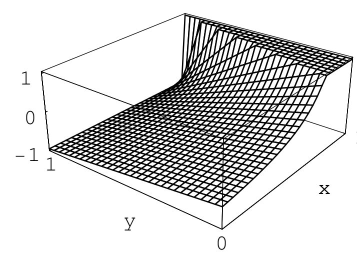

Figure 2.6: Plot of the function f from the definition of the FD procedure. Notice the discontinuity in $(1,1)$.

An important question that remains to be answered is how good the approximation $\tilde{\omega}_{\mathrm{aprx}}$ is. The answer is that, at least for large values of N, it is very good:

Lemma 2.3.2. *The approximation $\tilde{\omega}_{aprx}$ generated by an FD procedure converges against the frequency $\tilde{\omega}$ for $N \to \infty$ (we use the same notation as in the definition of that procedure).*

Proof. Lemma 2.3.1 tells us that

$$\left|c_1(\tilde{t}+\tau)\right|^2 = 1 - \beta' + \beta' \cos(\omega\tau),$$

where β' is defined by the eigensystem of the observed qubit's Hamiltonian. Using the abbreviations $a = \left|c_1(\tilde{t}+\frac{\Delta t}{2})\right|^2$, $b = \left|c_1(\tilde{t}+\Delta t)\right|^2$, and $c = \cos(\omega\tau)$, we get (recall that $\tau \in \{\frac{\Delta t}{2}, \Delta t\}$)

$$a = 1 - \beta' + \beta' c,\ b = 1 - \beta' + \beta'(2c^2 - 1).$$

With the frequency $\tilde{\omega}$ and the function f (see the definition of the FD procedure) it follows, after some algebraic manipulations, that

$\tilde{\omega} = \frac{2}{\Delta t} \arccos f(a, b)$. Mainly (a special case has to be considered for $a = 1, b = 1$) with the observations that

$$\lim_{N\to\infty} \tfrac{M(\tau)}{N} = \left|c_1(\tilde{t} + \tau)\right|^2 = \begin{cases} a & \text{for } \tau = \Delta t/2, \\ b & \text{for } \tau = \Delta t \end{cases}$$

and that f is a continuous function on $[0, 1] \times [0, 1] \backslash (1, 1)$ (see figure 2.6) it follows that

$$\tilde{\omega} = \frac{2}{\Delta t} \lim_{N\to\infty} \arccos f(a_{\mathrm{aprx}}, b_{\mathrm{aprx}}),$$

where a_{aprx} and b_{aprx} are defined as in the definition of the FD procedure. □

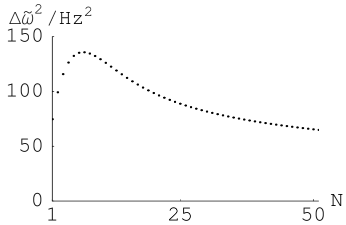

Figure 2.7: Plot of the mean square deviation $\Delta\tilde{\omega}^2$ for $\tilde{\omega} = 2\pi$ Hz, $\beta' = 1/3$, $\Delta t = 0.18\Delta t_{\max}$, and $N \in \{1, 2, \ldots, 51\}$.

Now that we know that $\tilde{\omega}_{\mathrm{aprx}}$ converges against $\tilde{\omega}$ with rising N, another interesting question arises: How should the parameter Δt be chosen so that the convergence is quick? A good method to answer this question is to investigate the mean square deviation of $\tilde{\omega}_{\mathrm{aprx}}$ from $\tilde{\omega}$ for different values of $N$and Δt. It is

$$\Delta\tilde{\omega}^2(N, \Delta t) = \sum_{j=0}^{N} \sum_{k=0}^{N} p(j, k, N, \Delta t) \Big(\tilde{\omega}_{\mathrm{aprx}} \Big|_{M(\Delta t/2)=j, M(\Delta t)=k} - \tilde{\omega} \Big)^2,$$

where $p(j, k, N, \Delta t)$ is the probability for a sequence of measurement results with $M(\frac{\Delta t}{2}) = j$ and $M(\Delta t) = k$. Using basic probability theory, this quantity can be converted into an expression that is straight forward to evaluate:

$$\Delta\tilde{\omega}^2(N, \Delta t) =$$
$$\sum_{j=0}^{N} p(j, N, \frac{\Delta t}{2}) \sum_{k=0}^{N} p(k, N, \Delta t) \Big(\tilde{\omega}_{\mathrm{aprx}} \Big|_{M(\Delta t/2)=j, M(\Delta t)=k} - \tilde{\omega} \Big)^2,$$

where

$$p(j, N, \tau) = \binom{N}{j} \begin{cases} 1 & \text{if } (p_1(\tau) = 0 \wedge j = 0) \\ & \text{or } (p_1(\tau) = 1 \wedge j = N), \\ p_1(\tau)^j \big(1 - p_1(\tau)\big)^{N-j} & \text{otherwise.} \end{cases}$$

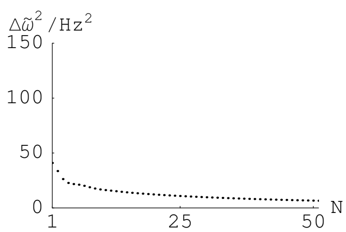

Figure 2.8: Plot of the mean square deviation $\Delta\tilde{\omega}^2$ for $\tilde{\omega} = 2\pi$ Hz, $\beta' = 1/3$, $\Delta t = 0.4\Delta t_{\max}$, and $N \in \{1, 2, \ldots, 51\}$.

with $p_1(\tau) = \left|c_1(\tilde{t} + \tau)\right|^2$.

Figure 2.7 shows a plot of $\Delta\tilde{\omega}^2(N, \Delta t)$ for $\beta' = 1/3$ (recall from lemma 2.3.1 that $\left|c_1(\tilde{t} + \tau)\right|^2$, and thus $p_1(\tau)$, depends on β') and a value of Δt that is small as compared to $\Delta t_{\max}$. What is surprising is the fact that, with rising N, the deviation first gets larger before it eventually converges against zero (which it always does, according to lemma 2.3.2). An explanation for this peculiar behavior may be that, because Δt is small, the parameter $|c_1|^2$ hardly evolves in between

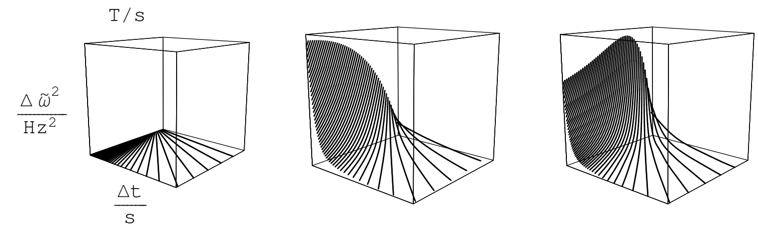

Figure 2.10: Plot from figure 2.9 for different values of β': $\beta' = 0$ (left plot), $\beta' = \frac{1}{6}$ (middle plot), $\beta' = \frac{1}{2}$ (right plot).

measurements. Consequently, the pair $(a_{\text{aprx}}, b_{\text{aprx}})$ is likely to be close to $(1, 1)$, and that's where f is discontinuous (see figure 2.6). This explanation is supported by figure 2.8. Here, Δt is larger and the above effect does not occur. When comparing figures 2.7 and 2.8, it appears that the convergence is the quicker, the closer Δt is to $\Delta t_{\max}$. This assumption is supported by figure 2.9 where the value of $\Delta\tilde{\omega}^2$ is plotted over the total run time $T = N\frac{\Delta t}{2} + N\Delta t = \frac{3}{2}N\Delta t$ of the FD procedure and over the parameter Δt. Figure 2.10 shows the same plot with different values for the amplitude β'. If $\beta' = 0$ the Hamiltonian rotates Bloch points about the z axis and, therefore, $\tilde{\omega} = 0$. In this case, the convergence behavior is always optimal.

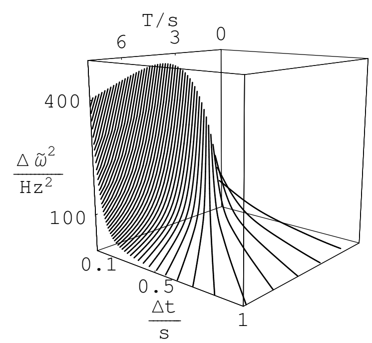

Figure 2.9: Plot of the mean square deviation $\Delta\tilde{\omega}^2$ for $\tilde{\omega} = 2\pi$ Hz, and $\beta' = 1/3$. The quantity $T = \frac{3}{2}N\Delta t$ is the total run time of an FD procedure. Each curve corresponds to one value of N.

What follows from the above observations? Answer: The convergence of $\tilde{\omega}_{\text{aprx}}$ towards $\tilde{\omega}$ is usually slow! The rationale is that a user of the FD procedure has to choose a small value for Δt, unless he knows the value of Δt in good approximation. Solving this problem by finding an alternative FD procedure could be an interesting topic for future research. What would be quite sensational—as all things that seem unlikely to be true—is a procedure with a convergence behavior similar to that shown in figure 2.11. Such a behavior would make it possible to determine $\tilde{\omega}$ in an, at least in theory, infinitesimally short amount of time, even if there is no a priory information about $\tilde{\omega}$. A recipe to find the procedure with the quickest convergence is easy to phrase but probably hard to execute:

> Given a fixed total measurement time, find the parameters (measurement operators, distances between measurements, etc.) that minimize $\Delta\tilde{\omega}^2$, averaged over all possible values of $\tilde{\omega}$, α, etc..

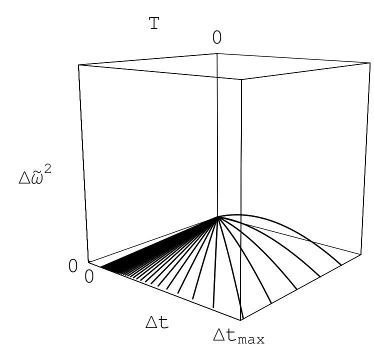

Figure 2.11: Plot of the mean square deviation $\Delta\tilde{\omega}^2$ for a fictive alternative FD procedure that shows very quick convergence for small values of Δt.

2.4 Frequency Tracking

Since the FD procedure provides a good way to get information about the dynamics of a qubit, it is suggestive to construct tracking procedures from it. For example, one could take an FD procedure and, instead of only once at the end, estimate the frequency $\tilde{\omega}$ continuously as measurement results come in. The current value of $\tilde{\omega}_{\text{aprx}}$

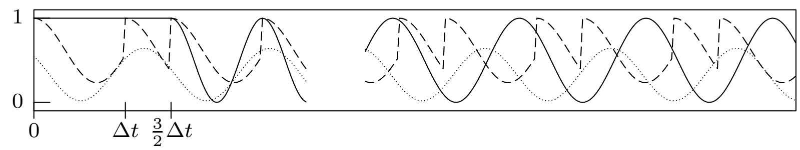

Figure 2.12: Evolution of the parameter $|c_1|^2$ (dashed curve), the undisturbed parameter $|\tilde{c}_1|^2$ (dotted curve), and the guess g_1 (solid curve) under the influence of an IFT procedure with $\beta'_g = 1/2$ (simulated with Trasim). The left side of the plot shows the evolution at the beginning, the right side shows the evolution at a later time, where $\tilde{\omega}_{\text{aprx}}$ is already a good approximation of $\tilde{\omega}$.

could then be used as a basis for a guess $g_1(t)$ of $|c_1(t)|^2$. What follows is a precise definition of just this procedure.

CFT Procedure.
Full name: Continuous Frequency Tracking Procedure.
Definitions: Same as for FT procedure.
Parameters: Same as for FT procedure plus the amplitude β'_g.
Steps: Same as for the FT procedure, except for the following differences:

- The order in which the time intervals Δt and $\frac{\Delta t}{2}$ are assigned to τ does matter. It should be $\frac{\Delta t}{2}$, Δt, $\frac{\Delta t}{2}$, Δt, ..., or Δt, $\frac{\Delta t}{2}$, Δt, $\frac{\Delta t}{2}$, ...
- After every two measurements in the main loop, a variable j is incremented by one. Initially it is set to zero.
- After every two steps in the main loop, and after j has been incremented, the approximation $\tilde{\omega}_{\text{aprx}}$ is updated using $a_{\text{aprx}} = M(\frac{\Delta t}{2})/j$ and $b_{\text{aprx}} = M(\Delta t)/j$. Initially, $\tilde{\omega}_{\text{aprx}}$ is set to zero.
- $|c_1|^2$ is continuously approximated by

$$g_1(t) = 1 - \beta'_g + \beta'_g \cos\Big(\int_{t_0}^{t} dt'\, \tilde{\omega}_{\text{aprx}}(t')\Big),$$

  where t_0 is the time when tracking started.

Figure 2.12 displays a simulated run of the CFT procedure. We see that, although the frequency of the guessed evolution $g_1(t)$ eventually matches that of the undisturbed evolution $|\tilde{c}_1(t)|^2$ quite closely, the guess does not serve its actual purpose very well: It does not reflect the dynamics of $|c_1(t)|^2$. Therefore, the deviation of this procedure is very high. In addition, due to the frequent projective measurements, the disturbance is very high as well. We conclude that the CFT procedure is not very attractive for the purpose of tracking.

Let us turn to another tracking procedure that is also based on the FD procedure:

IFT Procedure.
Full name: Initial Frequency Tracking Procedure.

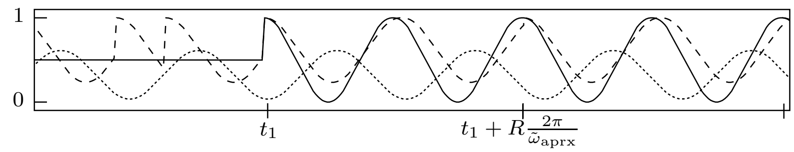

Figure 2.13: Evolution of the parameter $|c_1|^2$ (dashed curve), the undisturbed parameter $|\tilde{c}_1|^2$ (dotted curve), and the guess g_1 (solid curve) under the influence of an IFT procedure with $\beta'_g = 1/2$ (not simulated—drawn by hand). More specifically, only the evolution in the vicinity of t_1 is displayed.

Definitions: Same as for FT procedure.
Parameters: Same as for FT procedure plus the amplitude β'_g, the total tracking duration $T > \frac{3}{2}N\Delta t$, and the reset interval $R \in \{1, 2, 3, \ldots\}$.
Steps:
1. The value $\tilde{\omega}_{\text{aprx}}$ is determined using the FD procedure. During this process, the parameter $|c_1(t)|^2$ is approximated by $g_1(t) = \frac{1}{2}$.
2. By measurement, the state vector $|\psi\rangle$ is transformed into $|1\rangle$.
3. $|c_1(t)|^2$ is approximated by

$$g_1(t) = 1 - \beta'_g + \beta'_g \cos\bigl(\tilde{\omega}_{\text{aprx}}(t - t_1)\bigr),$$

where t_1 is the time when the FD procedure finished. Every R periods of duration $2\pi/\tilde{\omega}_{\text{aprx}}$, the quantity $|c_1(t)|^2$ is reset to $|1\rangle$.

A fictive run of the IFT procedure is visualized in figure 2.13. We see that, after the frequency determination process has finished, the curves corresponding to $g_1(t)$ and to $|c_1(t)|^2$ are quite close. One reason, of course, is that, at this point, the guess approximately "knows" the frequency $\tilde{\omega}$. The other reason is that, through repeated resets at times t_1, $t_1 + R\frac{2\pi}{\tilde{\omega}_{\text{aprx}}}$, $t_1 + 2R\frac{2\pi}{\tilde{\omega}_{\text{aprx}}}$, an so on, both curves are kept "in sync". It follows that the tracking deviation of the IFT procedure is relatively small for long tracking durations where the impact of the bad guess during the initial process of frequency determination can be neglected. In fact, the tracking deviation could be brought close to zero by replacing the parameter β'_g by a close estimation for the amplitude β'. Such an estimation should be possible to get because, as we know from lemma 2.3.1, the value of β' depends only on the Hamiltonian of the system under observation.

Although the IFT procedure, at first sight, looks like a very nice tracking procedure, it has considerable disadvantages:

- The use of projective measurements causes a strong disruption of the original dynamics of the qubit under observation. Consequently, the disturbance is very high, even for short tracking durations.
- The procedure is unstable: The guess does not adapt to (slight)

changes in the qubit's Hamiltonian that may happen under "real world conditions".

3 Estimation and Tracking with Minimal Measurements

In this chapter we discuss tracking and postestimation procedures using minimal measurements. We recall from chapter 1 that minimal measurements are measurements with positive measurement operators. Note that, in general, finding optimal (or superoptimal) parameters for postestimation procedures utilizing minimal measurements is harder than for procedures utilizing only projective measurements. The reason is that minimal measurements are a superset of projective measurements and, therefore, have more parameters.

3.1 Postestimation

Let us define an extension of the SPMP procedure defined in the last chapter to minimal measurements:

SMMP Procedure.
Full name: Single Minimal Measurement Postestimation Procedure.
Parameters:

- A set of minimal measurement operators $M_0, M_1, \ldots, M_{N-1}$.
- A guess function g_1.

Steps:

1. The state ψ of the observed qubit is measured using the measurement operators $M_0, M_1, \ldots, M_{N-1}$. The outcome is denoted by m, and the postmeasurement state by ψ'.
2. The parameters $|c_1'|^2 = |\langle 1|\psi'\rangle|^2$ is estimated by the guess $g_1(m)$.

Finding all optimal parameters of this procedure for the general case of an arbitrary number of measurement operators seems far too complicated. Therefore, we will limit ourselves to the special case of two measurement operators, below. For the general case, it is however possible to specify the optimal guess (under the common side condition that the initial state of the qubit under observation is totally unknown):

Lemma 3.1.1. *Under the side condition that the initial state of the observed qubit is totally unknown, the optimal guess for the SMMP procedure (measurement operators: $M_0, M_1, \ldots, M_{(N-1)}$, $N \geq 0$) is*

$$g_1'(m) = \frac{\langle 1| M_m M_m^\dagger |1\rangle}{\langle 0| M_m^\dagger M_m |0\rangle + \langle 1| M_m^\dagger M_m |1\rangle}$$

for all m with $M_m \neq 0$ (specifying a guess for the m with $M_m = 0$ is pointless since the probability to get such an m as measurement result is zero).

Proof. The guess results from minimizing the postestimation deviation

$$v'[g_1'(m)] = \int d\varrho(\psi) \sum_{m':M_{m'}=0} p(m'|\psi)\big(g_1'(m') - |c_1'(m')|^2\big)^2,$$

where $p(m'|\psi)$ is the probability to measure m', and $\varrho(\psi)$ is the uniform probability density of the initial states. □

3.1.1 Postestimation with Two Outcomes

During the investigation of an SMMP procedure with two possible outcomes, it is convenient to parameterize the measurement operators according to the following decomposition:

Lemma 3.1.2. *If $\{M_0, M_1\}$ is a set of measurement operators describing a minimal measurement on a qubit, then the following decomposition is possible:*

$$M_0 = \sqrt{p_0}\,|p_0\rangle\langle p_0| + \sqrt{p_1}\,|p_1\rangle\langle p_1|,$$
$$M_1 = \sqrt{1-p_0}\,|p_0\rangle\langle p_0| + \sqrt{1-p_1}\,|p_1\rangle\langle p_1|,$$

where $p_j \in [0,1]$ ($j \in \{0,1\}$) and $\langle p_0|p_1\rangle = 0$.

Proof. By definition the operators M_0 and M_1 are positive. Therefore, they can be spectrally decomposed into

$$M_0 = a_0\,|p_0\rangle\langle p_0| + a_1\,|p_1\rangle\langle p_1|,$$
$$M_1 = b_0\,|q_0\rangle\langle q_0| + b_1\,|q_1\rangle\langle q_1|,$$

where the coefficients a_j, b_j ($j \in \{0,1\}$) are greater or equal to zero, and the vectors $|p_j\rangle$, $|q_j\rangle$ satisfy the orthogonality conditions $\langle p_0|p_1\rangle = 0$ and $\langle q_0|q_1\rangle = 0$. With the completeness relation $M_0^\dagger M_0 + M_1^\dagger M_1 = \mathbb{1}$ and elementary algebraic manipulations it follows that the conjectured decomposition is possible. □

Using the above parameterization, we can define the promised SMMP procedure:

SMMP2 Procedure.
Full name: Single Minimal Measurement Postestimation Procedure with two Outcomes.
Parameters:

- A set of parameters $p_j \in [0,1]$ ($j \in \{0,1\}$) and $\langle p_0|p_1\rangle = 0$ characterizing the measurement operators.
- A guess g_1'.

Steps:

1. The state ψ of the qubit under observation is measured using the measurement operators

$$M_m = \sqrt{\alpha_0^m} \left| p_0 \right\rangle \left\langle p_0 \right| + \sqrt{\alpha_1^m} \left| p_1 \right\rangle \left\langle p_1 \right|,$$

   where $\alpha_j^0 = p_j$, and $\alpha_j^1 = 1 - p_j$ ($j \in \{0, 1\}$). The post measurement state of the qubit is denoted by ψ_m.
2. The parameter parameter $\left| c_1'(m) \right|^2 = \left| \langle 1 | \psi_m' \rangle \right|^2$ is estimated, using the guess g_1'.

The optimal guess for the SMMP2 procedure can be specified in a small and compact form:

Theorem 3.1.1. *If the initial state of the observed qubit is totally unknown, then the optimal guess for the SMMP2 procedure is*

$$g_1'(m) = \frac{\alpha_0^m \left| \beta_0 \right|^2 + \alpha_1^m \left| \beta_1 \right|^2}{\alpha_0^m + \alpha_1^m}, \tag{3.1.1}$$

where $\beta_j = \langle 1 | p_j \rangle$, and the parameters α_j^m and $\left| p_j \right\rangle$ are defined as in the definition of that procedure.

Proof. Equation (3.1.1) follows by insertion of the measurement operators defined by α_j^m and $\left| p_j \right\rangle$ into the best guess according to lemma 3.1.1. □

What follows is the disturbance and deviation of the SMMP2 procedure with optimal guess, unfortunately not in a small and compact form. As mentioned below, the presented expressions are not applicable to all possible parameter combinations. Because the missing combinations represent only a small set of special cases, and because it is reasonable to expect that it is sufficient to know values in their vicinity (i.e. we assume continuity), we dare to exclude these from further discussion.[1]

Lemma 3.1.3. *If the initial state of the observed system is totally unknown, if the optimal guess is used, and if $p_j \notin \{0, 1, p_{j-1}\}$ ($j \in \{0, 1\}$), then the disturbance and deviation of the SMMP2 procedure*

[1] Note that when discussing the generalization of the SMMP2 procedures to general measurements in the next chapter, no parameter combinations are excluded.

are

$$
\begin{aligned}
s = \frac{1}{12} \sum_{m\in\{0,1\}} \Bigg(& \alpha_0^m \left(1 + 2\,|\beta_0|^2\right) + \alpha_1^m \left(3 - 2\,|\beta_0|^2\right) \\
& + \frac{6}{(\alpha_0^m - \alpha_1^m)^3} \Bigg(\alpha_0^{m\,4}\,|\beta_0|^2 - 4\alpha_0^{m\,3}\alpha_1^m\,|\beta_0|^2 \left(2\,|\beta_0|^2 - 1\right) \\
& + 4\alpha_0^m \alpha_1^{m\,3} \left(|\beta_0|^2 - 1\right)\left(2\,|\beta_0|^2 - 1\right) - \alpha_1^{m\,4} \left(|\beta_0|^2 - 1\right)^2 \\
& + \alpha_0^{m\,2}\alpha_1^{m\,2} \Big(2 \ln \frac{\alpha_0^m}{\alpha_1^m} + 6\,|\beta_0|^2 \left(1 + 2\ln\frac{\alpha_0^m}{\alpha_1^m}(|\beta_0|^2 - 1)\right) - 3\Big)\Bigg) \\
& + 4\Big(2\sqrt{\alpha_0^m\alpha_1^m}\,|\beta_0|^2 \left(|\beta_0|^2 - 1\right) - \alpha_0^m\,|\beta_0|^2 \left(|\beta_0|^2 + 1\right) \\
& \qquad - \alpha_1^m \left(|\beta_0|^2 - 2\right)\left(|\beta_0|^2 - 1\right)\Big)\Bigg)
\end{aligned}
$$

and

$$
\begin{aligned}
v' = \sum_{m\in\{0,1\}} & \frac{\alpha_0^m\alpha_1^m}{(\alpha_0^m - \alpha_1^m)^3(\alpha_0^m - \alpha_1^m)} \sum_{j=0}^{1} \Bigg(\alpha_j^{m\,2}\alpha_{1-j}^m \Big(\ln\frac{\alpha_0^m}{\alpha_1^m} - 2(-1)^j \\
& + 3\left(2\ln\frac{\alpha_0^m}{\alpha_1^m} - 3(-1)^j\right)|\beta_0|^2 \left(|\beta_0|^2 - 1\right)\Big) - (-1)^j \alpha_j^{m\,3}\,|\beta_0|^2 \left(|\beta_0|^2 - 1\right)\Bigg),
\end{aligned}
$$

where $\beta_0 = \langle 1|p_0\rangle$, *and the parameters* p_j, α_j^m *and* $|p_j\rangle$ *are defined as in the definition of that procedure.*

Proof. According to their definition in chapter 1, the disturbance and deviation are for the the SMMP2 procedure

$$
\begin{aligned}
s &= \int d\psi\, \varrho(\psi) \textstyle\sum_{m\in\{0,1\}:\, M_m\neq 0} p(m|\psi)\left(|c_1'(m)|^2 - |c_1|^2\right)^2, \\
v' &= \int d\psi\, \varrho(\psi) \textstyle\sum_{m\in\{0,1\}:\, M_m\neq 0} p(m|\psi)\left(g_1 - |c_1'(m)|^2\right)^2,
\end{aligned}
$$

where $\varrho(\psi)$ is the probability density of the premeasurement states, and $p(m|\psi)$ is the probability to the result m, given an initial state ψ. The conjectured expressions follow by straight forward, albeit lengthy, calculation. □

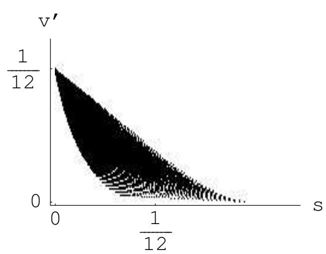

Figure 3.1: Numerical approximation of the disturbance deviation graph of the SMMP2 procedure.

Via the disturbance and deviation, it is possible to find the optimal and superoptimal parameters of the SMMP2 procedure. One way to accomplish this is to minimize the deviation under the side condition that the disturbance is fixed (and, additionally, the other way round, if the superoptimal parameters are desired). Such a calculation may be done analytically or numerically. Another method is to numerically sample over all possible parameter combinations, for each sample, calculate the disturbance and deviation. By comparing the resulting values an approximation for the optimal and superoptimal parameters can be obtained. This latter method is what we will now discuss.

Figure 3.1 shows a graph visualizing the disturbance and deviation corresponding to all parameter combinations $(p_0, p_1, |\beta_0|^2) \in \{0, 0.01, 0.02, \dots, 1\}^3$, except those that are excluded in lemma 3.1.3 and, because symmetry suggests it, those with $p_1 < p_0$ (this speeds up the data generation process). It is to be noted that in the lower right part the plot is very "thin", probably because the sampling grid wasn't fine enough. The displayed disturbance and deviation values were generated with a computer program that essentially comprises the following steps:

1. A table, we call it *SV table*, is created. Its columns correspond to disturbances, and its rows to deviations. For example column one may correspond to all disturbances $s \in [0, 0.1)$, column two to $s \in [0.1, 0.2)$, and so on. The cells in the table are initialized with empty lists.
2. For each sampled parameter set, the disturbance and deviation is calculated, and the set is appended to the list in the corresponding cell in the SV table.

The graph in figure 3.1 is a density plot where the black points correspond to nonempty lists in the SV table.

Once the table is filled, it is simple to get an approximation for the optimal (superoptimal) parameters of the SMMP2 procedure. All that needs to be done is to extract the bottommost (and leftmost) non empty cells and take the arithmetic mean of all the parameter sets contained in each such cell. The result is displayed in table 3.1 where, to ease interpretation of the results, the parameters p_0 and p_1 are encoded into $\bar{p} = \frac{1}{2}(p_0 + p_1)$ and $\Delta p = p_1 - p_0$. We see that $\bar{p}$ is equal to $\frac{1}{2}$ throughout. The same is true for $|\beta_0|^2$. This means that the parameter ϑ of

$$|p_0\rangle = e^{i\chi}\big(\cos\tfrac{\vartheta}{2}\,|0\rangle + e^{i\varphi}\sin\tfrac{\vartheta}{2}\,|1\rangle\big),\ \chi \in [0, 2\pi), \varphi \in [0, 2\pi), \vartheta \in [0, \pi]$$

satisfies $\cos\frac{\vartheta}{2} = 1/\sqrt{2}$. Consequently, $\vartheta = \frac{\pi}{2}$ and, thus, the Bloch point $P(p_0)$ lies on the equator of the Bloch sphere. With the help of lemma 2.0.2 we infer (recall that, due to symmetry considerations, any values with $p_1 < p_0$ were not sampled—now they must be included in the form of $\Delta p \in [-1, 0)$):

> The set of superoptimal parameters for the SMMP2 procedure comprises
>
> - all p_0, p_1 with $\bar{p} = \frac{1}{2}$ and $\Delta p \in [-1, 1]$,
> - all $|p_0\rangle$, $|p_1\rangle$, where $P(p_0)$ and $P(p_1)$ lie on diametrically opposing points on the Bloch sphere's equator (see figure 3.2).

The disturbance deviation graph for the SMMP2 procedure with these parameters is displayed in figure 3.3. As expected the it roughly corresponds to the bottom- and leftmost values of the density plot in figure 3.1.

By inserting the superoptimal parameters into the optimal guess (see lemma 3.1.1) we get a surprising result:

$\bar{p}$	Δp	$\lvert\beta_0\rvert^2$
0.50000	0.98000	0.50000
0.50000	0.98000	0.50000
0.50000	0.98000	0.50000
0.50000	0.98000	0.50000
0.50000	0.98000	0.50000
0.50000	0.97000	0.50000
0.50000	0.97000	0.50000
0.50000	0.96000	0.50000
0.50000	0.96000	0.50000
0.50000	0.95000	0.50000
0.50000	0.95000	0.50000
0.50000	0.94000	0.50000
0.50000	0.94000	0.50000
0.50000	0.93000	0.50000
0.50000	0.93000	0.50000
0.50000	0.92000	0.50000
0.50000	0.92000	0.50000
0.50000	0.91000	0.50000
0.50000	0.91000	0.50000
0.50000	0.90000	0.50000
0.50000	0.90000	0.50000
0.50000	0.89000	0.50000
0.50000	0.88000	0.50000
0.50000	0.88000	0.50000
0.50000	0.87000	0.50000
0.50000	0.87000	0.50000
0.50000	0.86000	0.50000
0.50000	0.85000	0.50000
0.50000	0.85000	0.50000
0.50000	0.84000	0.50000
0.50000	0.83000	0.50000
0.50000	0.83000	0.50000
0.50000	0.82000	0.50000
0.50000	0.81000	0.50000
0.50000	0.80727	0.50000
0.50000	0.80000	0.50000
0.50000	0.79000	0.50000
0.50000	0.78000	0.50000
0.50000	0.77662	0.50000
0.50000	0.77000	0.50000
0.50000	0.76000	0.50000
0.50000	0.75000	0.50000
0.50000	0.74000	0.50000
0.50000	0.73595	0.50000
0.50000	0.73000	0.50000
0.50000	0.71856	0.50000
0.50000	0.71000	0.50000
0.50000	0.69963	0.50000
0.50000	0.69000	0.50000
0.50000	0.67909	0.50000
0.50000	0.67000	0.50000
0.50000	0.66283	0.50000
0.50000	0.65081	0.50000
0.50000	0.64295	0.50000
0.50000	0.63122	0.50000
0.50000	0.62000	0.50000
0.50000	0.60977	0.50000
0.50000	0.60000	0.50000
0.50000	0.58841	0.50000
0.50000	0.57383	0.50000
0.50000	0.56647	0.50000
0.50000	0.55257	0.50000
0.50000	0.53686	0.50000
0.50000	0.52760	0.50000
0.50000	0.51193	0.50000
0.50000	0.49463	0.50000
0.50000	0.48640	0.50000
0.50000	0.46813	0.50000
0.50000	0.44776	0.50000
0.50000	0.44000	0.50000
0.50000	0.41919	0.50000
0.50000	0.39698	0.50000
0.50000	0.38802	0.50000
0.50000	0.36413	0.50000
0.50000	0.33740	0.50000
0.50000	0.32963	0.50000
0.50000	0.30036	0.50000
0.50000	0.26832	0.50000
0.50000	0.23600	0.50000
0.50000	0.21440	0.50000
0.50000	0.17805	0.50000
0.50000	0.13583	0.50000
0.50000	0.08176	0.50000
0.50000	0.03007	0.50000

Table 3.1: Numerical approximation of the superoptimal parameters of the SMMP2 procedure.

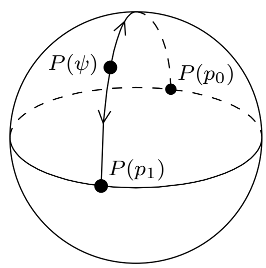

Figure 3.2: The Bloch points corresponding to the optimal parameters $|p_0\rangle$ and $|p_1\rangle$ lie on the equator of the Bloch sphere. Due to measurement, the Bloch point of the observed qubit moves on a great circle towards one of them.

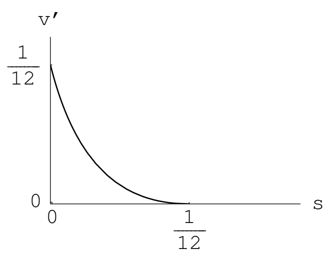

Figure 3.3: Disturbance deviation graph of the SMMP2 procedure with optimal parameters.

The superoptimal guess for the the SMMP2 is $g_1'(m) = \frac{1}{2}$.

To get some insight into why, although the guess is independent of the measurement readout, the deviation can, by variation of the parameters, be continuously lowered to zero (recall figure 3.3), let us investigate the two special cases:

- For $\Delta p = 1$ the measurement operators are projective: $M_0 = |p_1\rangle\langle p_1|$, $M_1 = |p_0\rangle\langle p_0|$. Thus, they transform the state vector ψ of the estimated qubit into $|p_0\rangle$ or $|p_1\rangle$. On the Bloch sphere this transformation can be visualized as a rotation of the Bloch point $P(\psi)$ onto the equator. The consequence is that $|c_1'|^2$ evaluates to $\frac{1}{2}$ and, thus, the deviation is zero (the corresponding disturbance, according to figure 3.3, is $\frac{1}{12}$).
- For $\Delta p = 0$ the measurement operators are proportional to the unit operator: $M_0 = \frac{1}{2}\mathbb{1}$, $M_1 = \frac{1}{2}\mathbb{1}$. Obviously, they do not transform $|\psi\rangle$ and the Bloch point $P(\psi)$ is not moved. Consequently, the disturbance is zero (the corresponding deviation is $\frac{1}{12}$).

We infer that all other possible disturbance deviation displayed in figure 3.3 can be achieved by varying Δt.

The result that it is best to always choose 1/2 as a guess is, of course, not satisfying. An important question arises: "Is this guess also the best guess for estimation procedures that are not limited to minimal measurements?". As we will see in the next chapter, this question, fortunately, has to be answered in the negative.

3.2 N-Series Tracking

In this section we will touch a topic that has been discussed intensely by Audretsch et. al. [AKS01]. It is roughly based on the following tracking procedure which was designed for observing the evolution of the parameter $|c_1(t)|^2$ of a qubit with Hamiltonian

$$H_{\mathrm{I}} = \hbar D(|1\rangle\langle 0| + |0\rangle\langle 1|),$$

where the factor $D > 0$ controls the frequency of the evolution.

NST Procedure.
Full name: N-Series Tracking Procedure.
Parameters:
A set of parameters $p_j \in [0,1]$ ($j \in \{0,1\}$) and $\langle p_0|p_1\rangle = 0$ characterizing the measurement operators.
The temporal distance between measurements, τ.
The guess $g_1(t)$ for the parameter $|c_1(t)|^2$ of the observed system.
The number of measurements per N-series, $N >= 1$.
The number of N-series to perform, Z.
Steps: The state $\psi(t)$ of the qubit is sequentially being measured using the measurement operators

$$M_m = \sqrt{\alpha_0^m}\,|0\rangle\langle 0| + \sqrt{\alpha_1^m}\,|1\rangle\langle 1|,$$

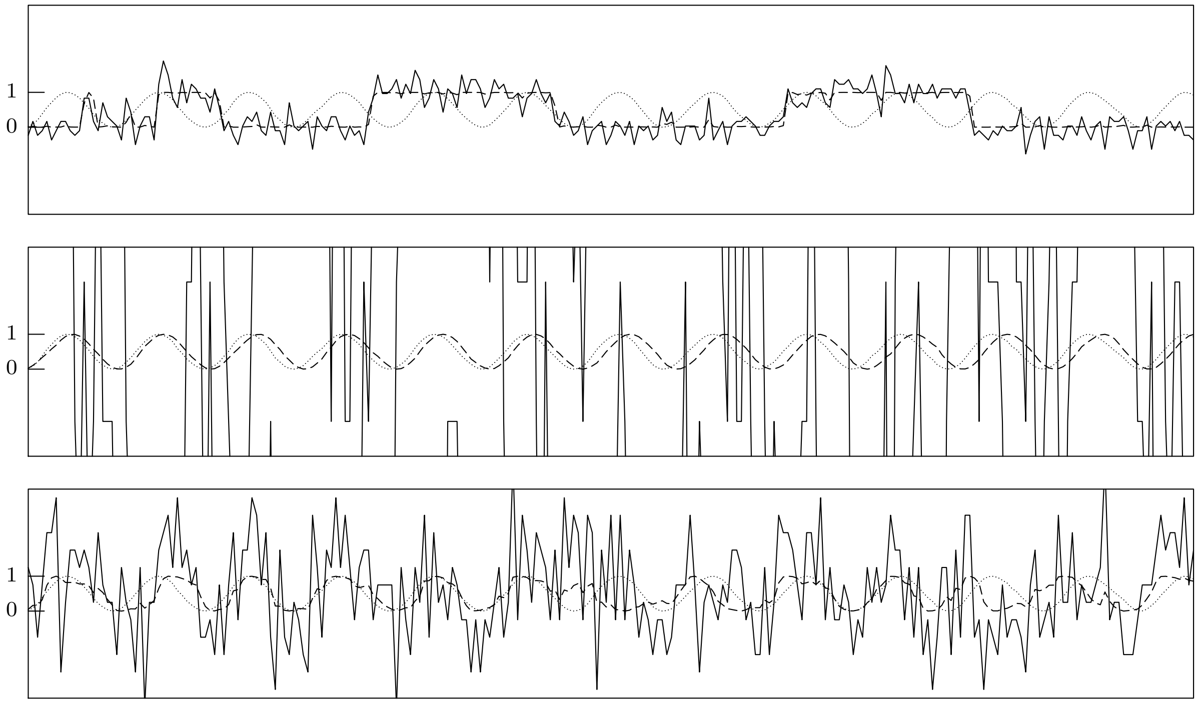

Figure 3.4: Simulation of the FST procedure applied to a qubit with a Hamiltonian of the form $H_{\mathrm{I}} = \hbar D(|1\rangle\langle 0| + |0\rangle\langle 1|)$, $D > 0$ (generated from data that Audretsch et al. used for plots in a publication [AKS01]). The parameters used are $\bar{p} = \frac{1}{2}(p_0 + p_1) = 1/2$ and $\Delta p = p_1 - p_0 = -0.3$ (top plot), $\Delta p = 0.01$ (middle plot), $\Delta p = 0.08$ (bottom plot). The dotted curve corresponds to the undisturbed evolution (i.e. without measurements) of the qubit's parameter $|c_1|^2$, the dashed curve to actual evolution of $|c_1|^2$, and the solid one to the guess $g_1(N_0, N) = (r - p_0)/(\Delta p)$.

where $\alpha_j^0 = p_j$, and $\alpha_j^1 = 1 - p_j$ ($j \in \{0, 1\}$). The temporal distance between individual measurements is τ and the total number of measurements performed is $N\,Z$. For each *N-series* (i.e. each subsequent set of N measurements) the number of measurement outcomes with readout $m = 0$ is counted and stored in the variable N_0. This variable is used to update the guess $g_1(t)$ once after each N-series.

A guess was proposed and motivated by Audretsch et. al. [AKS01]:

$$g_1(N_0, N) = \frac{r - p_0}{\Delta p},$$

where $r = N_0/N$ and, as in the previous section, $\Delta p = p_1 - p_0$. Three simulated runs of the NST procedure with this guess are displayed in figure 3.4. The measurements used for the top plot are almost projective. Therefore the dynamics of $|c_1|^2$ is closely matched by the guess. The price is, of course, a high disturbance. In the middle plot, the measurements operators are almost unit operators and thus, the disturbance is low but the guess is far off. What is interesting, is the bottom plot. Here, the evolution of $|c_1|^2$ is moderately disturbed and it is still possible to somewhat deduce its dynamics by observing the guess. To increase the quality of the guess, Audretsch et. al. proposed post processing by means of a Wiener filter and a cutoff filter (it cuts of small frequencies in the Fourier spectrum of the guess). Some words on these filters:

- Since tracking is, by definition, a real time process, the filters also need to be applied in real time. Therefore, it may take some amount of time until the filtered information becomes good.
- In general, the Wiener filter and the cutoff filter require their user to decide what information is relevant and which information is to be regarded as noise. It is hard, if not impossible, to automate this process.

It should be noted that, by discarding "noise", the filters also discard information.

Let us discuss an alternative guess that is based on our concept of disturbance and deviation (note that this guess, as stated below, cannot be used in conjunction with all possible measurement operators):

Lemma 3.2.1. *If the initial state of the observed qubit is completely unknown, if $p_0 \notin \{0,1\}$, and if $p_1 \notin \{0,1\}$, then the optimal guess for the NST procedure is*

$$g_1(N_0, N) = \frac{p_1^{N_0}(1-p_1)^{N-N_0}}{p_0^{N_0}(1-p_0)^{N-N_0} + p_1^{N_0}(1-p_1)^{N-N_0}}, \tag{3.2.1}$$

where N_0 is the number of measurement outcomes with readout $m = 0$ in the last N-series, and the parameters p_j and N are defined as in the definition of that procedure.

Proof. It can be shown [AKS01] that the application of N measurements in an N-series equals a single measurement with $N+1$ possible outcomes $N_1 \in \{0,1,2,\ldots,N\}$. The measurement operator corresponding to outcome N_1 is

$$\tilde{M}(N_0, N) = \sqrt{\binom{N}{N_0}} M_0^{N_0} M_1^{(N-N_0)}.$$

Equation (3.2.1) follows by insertion of this operator into the optimal guess according to lemma 3.1.1. □

The result of simulations of the NST procedure with that guess is displayed in figure 3.5. We see that, although the disturbance and deviation are low, the dynamics of the system cannot be deduced

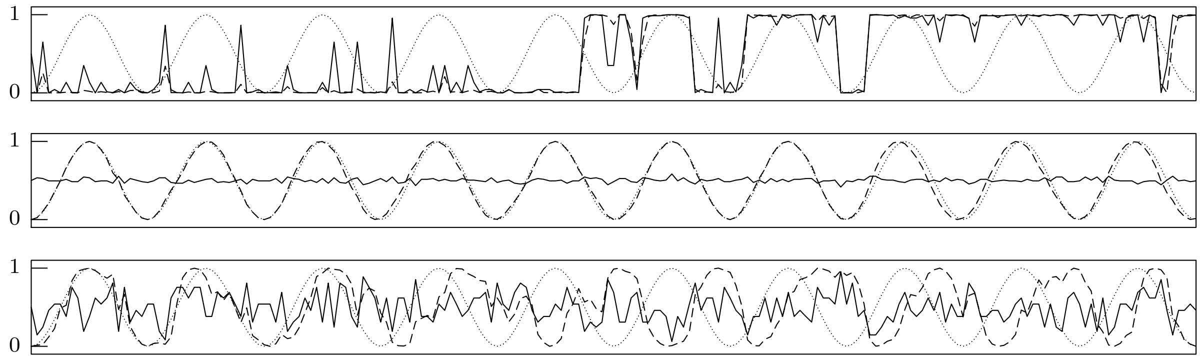

Figure 3.5: Simulation of the FST procedure with the same parameters as in figure 3.4, except for the guess $g_1(N_0, N)$ which is calculated according to lemma 3.2.1 (simulated with Trasim).

by looking at the curve corresponding to that guess. What we learn from this result is that disturbance and deviation are not necessarily good measures for gauging the quality of a tracking procedure.

4 Estimation and Tracking with General Measurements

The term general measurements refers to measurements that have no restrictions, i.e. all measurements according to the Measurement Postulate. This means that in the postestimation and tracking procedures that we discuss in this chapter, the measurements usually can be chosen at will, given that they satisfy minor side conditions such as a maximal number of measurement operators. This also means that finding the optimal or superoptimal parameters of those procedures is tedious, because there are so many of them.

4.1 Non-Sequential Postestimation

The term non-sequential postestimation refers to postestimation procedures that utilize only a single measurement instead of a sequence of measurements. In particular, we use the following definition:

SMP Procedure.
Full name: Single Measurement Postestimation Procedure.
Parameters:

- A set of measurement operators $M_0, M_1, \ldots, M_{N-1}$.
- A guess function g_1.

Steps:

1. The state vector $|\psi\rangle$ of the observed qubit is measured using the measurement operators $M_0, M_1, \ldots, M_{N-1}$. The readout is denoted by m and the postmeasurement state by $|\psi'_m\rangle$.
2. $|c'_1(m)|^2 = |\langle 1|\psi'_m\rangle|^2$ is estimated as g_1.

To find all the optimal parameters for this procedure for an arbitrary number of measurement operators is probably quite complicated. Therefore, we will discuss this problem only for the case of one and two measurement operators. It is, however, possible to specify the optimal guess for the general case:

Lemma 4.1.1. *Under the side condition that the initial state of the observed qubit is completely unknown, the optimal guess for the SMP procedure is*

$$g'_1(m) = \frac{\langle 1| M_m M_m^\dagger |1\rangle}{\langle 0| M_m^\dagger M_m |0\rangle + \langle 1| M_m^\dagger M_m |1\rangle}$$

for all m with $M_m \neq 0$.

Proof. The proof is identical to the proof of lemma 3.1.1 (optimal guess of the SMMP procedure) with the only difference that all references to operators of minimal measurements should be replaced by references to operators of general measurements. □

4.1.1 Postestimation with One Outcome

For the sake of completeness, before discussing more complicated SMP procedures, we start with a truly simple case:

SMP1 Procedure.

Full name: Single Measurement Postestimation Procedure with One Possible Outcome.

Parameters:

- A measurement operator M.
- A guess function g_1.

Steps:

1. The state vector $|\psi\rangle$ of the observed qubit is measured using the measurement operator M. The post measurement state is denoted by $|\psi'\rangle$.
2. $|c_1'|^2 = |\langle 1|\psi'\rangle|^2$ is estimated.

The optimal parameters may be obvious:

Lemma 4.1.2. *The optimal parameters for an SMP1 procedure are*

$$g_1' = \frac{1}{2},$$
$$M = U,$$

where U can be any unitary operator.

Proof. The value of the optimal guess follows by minimizing the deviation $v'(g_1')$ which, as can be shown, is equal to ${g_1'}^2 - g_1' + 1/3$. We see that the deviation does not depend on the measurement operator M. Therefore the choice of M does not matter, i.e. the parameters are always optimal as long as $g_1' = 1/2$. Certainly M needs to satisfy the completeness relation $M^\dagger M = \mathbb{1}$ that is required by the Measurement Postulate. Therefore it must be unitary. □

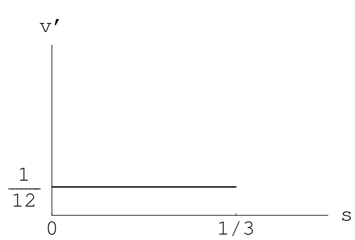

Figure 4.1: Disturbance-deviation graph of the SMP1 procedure with optimal parameters.

With the optimal parameters, the procedure's disturbance s and deviation v' are, proof being omitted, $s = \frac{1}{3}(1 - |\langle 1|U|1\rangle|^2)$ and $v' = 1/12$. Their knowledge allows the creation of a disturbance-deviation graph as displayed in figure 4.1. We see that, by bad choice of the measurement operator, the parameter $|c_1|^2$ can be considerably disturbed.

4.1.2 Postestimation with Two Outcomes

After the trivial case in the last subsection, let us turn to something more interesting:

SMP2 Procedure.

Full name: Single Measurement Postestimation Procedure with Two Outcomes.
Parameters:
- Two measurement operators M_0, M_1.
- A guess function g_1.

Steps:
1. The state vector $|\psi\rangle$ of the observed qubit is measured using the measurement operators M_0, M_1. The readout is denoted by m and the postmeasurement state by $|\psi'_m\rangle$.
2. $|c'_1(m)|^2 = |\langle 1|\psi'_m\rangle|^2$ is estimated by the guess g_1.

The best guess of that procedure is, of course, the one we found in lemma 4.1.1. Using it, we are able to determine the set of all *superoptimal* measurement operators (note that the determination of the set of all *optimal* operators may be more tedious because it is likely to be larger than the set of all superoptimal operators). The result, unfortunately, is very complicated:

Lemma 4.1.3. *Let*

$$a_0 \in [0,1], b_0 \in [0,1], \beta_m \in [0,1], \gamma_0 \in [0,1],$$
$$a_1, b_1, \gamma_1 : \begin{cases} (a_1, b_1, \gamma_1) \in \{(\sqrt{1-a_0^2}, \sqrt{1-b_0^2}, \gamma_0), \\ \qquad\qquad (\sqrt{1-b_0^2}, \sqrt{1-a_0^2}, 1-\gamma_0)\} & \textit{for } a_0 \neq b_0, \\ a_1 = \sqrt{1-a_0^2}, b_1 = \sqrt{1-a_0^2}, \gamma_1 \in [0,1] & \textit{for } a_0 = b_0, \end{cases}$$

be a set of parameters that satisfy the following condition: There is no other such set of parameters (distinguished by a tilde) with

$$s(\tilde\beta_0, \tilde\beta_1, \tilde a_0, \tilde a_1, \tilde b_0, \tilde b_1, \tilde\gamma_0, \tilde\gamma_1) = s(\beta_0, \beta_1, a_0, a_1, b_0, b_1, \gamma_0, \gamma_1) \quad \wedge$$
$$v'(\tilde\beta_0, \tilde\beta_1, \tilde a_0, \tilde a_1, \tilde b_0, \tilde b_1, \tilde\gamma_0, \tilde\gamma_1) < v'(\beta_0, \beta_1, a_0, a_1, b_0, b_1, \gamma_0, \gamma_1)$$

or with

$$v'(\tilde\beta_0, \tilde\beta_1, \tilde a_0, \tilde a_1, \tilde b_0, \tilde b_1, \tilde\gamma_0, \tilde\gamma_1) = v'(\beta_0, \beta_1, a_0, a_1, b_0, b_1, \gamma_0, \gamma_1) \quad \wedge$$
$$s(\tilde\beta_0, \tilde\beta_1, \tilde a_0, \tilde a_1, \tilde b_0, \tilde b_1, \tilde\gamma_0, \tilde\gamma_1) < s(\beta_0, \beta_1, a_0, a_1, b_0, b_1, \gamma_0, \gamma_1)$$

where the disturbance and postestimation deviation are

$$s = \sum_m P_m - \frac{1}{3}\sum_m \big(a_m^2\beta_m(1+\gamma_m) + b_m^2(1-\beta_m)(2-\gamma_m) + 2a_m b_m\sqrt{\beta_m(1-\beta_m)}\sqrt{\gamma_m(1-\gamma_m)}\big) + \frac{1}{3},$$

$$v' = \sum_m P_m - \frac{1}{2}\sum_m \begin{cases} 0 & \text{for } a_m = b_m = 0, \\ \frac{(a_m^2\beta_m + b_m^2(1-\beta_m))^2}{a_m^2+b_m^2} & \text{otherwise,} \end{cases}$$

$$P_m = \begin{cases} \frac{1}{3}a_m^3 & \text{for } a_m{=}b_m, \\ \frac{1}{2}b_m^2(\beta_m-1)^2 & \text{for } a_m{\neq}b_m, a_m{=}0, \\ \frac{1}{2}a_m^2\beta_m^2 & \text{for } a_m{\neq}b_m, b_m{=}0, \\ \frac{1}{2(a_m^2-b_m^2)^2}\big(a_m^6\beta_m^2 + b_m^6(1-\beta_m)^2 & \\ +a_m^4 b_m^2\beta_m(4-7\beta_m) & \\ +a_m^2 b_m^4(1-\beta_m)(7\beta_m-3)\big) & \\ +\frac{2a_m^4 b_m^4}{(a_m^2-b_m^2)^3}\big(1-6\beta_m(1-\beta_m)\big)\log\frac{a_m}{b_m} & \text{otherwise.} \end{cases}$$

Then the measurement operators M_0, M_1 that make an SMP2 procedure superoptimal *(under the side condition that any observed qubit is completely unknown) are operators of the form*

$$M_m = U_m D_m V_m,$$

where D_m, V_m, and U_m are defined as follows (note that, for any parameter combination, there are always *operators that satisfy the conditions below):*

- $D_m = a_m |0\rangle\langle 0| + b_m |1\rangle\langle 1|$.
- *V_m is a unitary operator with $|\langle 0| V_m |1\rangle|^2 = \gamma_m$ and, in the case that $a_0 \neq b_0 \wedge \gamma_0 \notin \{0,1\}$, additionally*

$$\arg\big((a_0^2-b_0^2)\langle 0| V_0^\dagger |0\rangle\langle 0| V_0 |1\rangle\big) = \arg\big(-(a_1^2-b_1^2)\langle 0| V_1^\dagger |0\rangle\langle 0| V_1 |1\rangle\big).$$

- *U_m is a unitary operator with $|\langle 0|\chi_m\rangle|^2 = \beta_m$ and*

$$\big(-\arg\langle 0|\chi_m\rangle + \arg\langle 1|\chi_m\rangle + \arg\langle 0|\xi_m\rangle - \arg\langle 1|\xi_m\rangle\big) \mod 2\pi = 0,$$

  where $|\chi_m\rangle = U_m^\dagger |1\rangle$ and $|\xi_m\rangle = V_m |1\rangle$.

Proof. The starting point of the proof follows from the definition of optimality. Initially it is required that the operators M_0, M_1 and the guesses g_0', g_1' satisfy the following condition:

There are no other operators $\tilde{M}_0, \tilde{M}_1$ and guesses $\tilde{g}_0', \tilde{g}_1'$ such that

$$s(\tilde{M}_0, \tilde{M}_0, \tilde{g}_0', \tilde{g}_1') = s(M_0, M_0, g_0', g_1') \quad \wedge$$
$$v'(\tilde{M}_0, \tilde{M}_0, \tilde{g}_0', \tilde{g}_1') < v'(M_0, M_0, g_0', g_1'),$$

where s and v' are the disturbance and postestimation deviation, respectively.

The optimal guess can, as we have seen before, be calculated by minimizing v'. After inserting them the above condition converts into a condition on just M_0 and M_1. The next step is to parameterize these operators so that the any integrals in s and v' over the qubit's initial state vector can be solved. It turns out that a good strategy is to decompose the operators into a singular value decomposition and parameterize the corresponding operators U_m, D_m, V_m using real values. After solving the integrals, more parameters can be eliminated, also by the introduction of the additional condition for superoptimality. The result is the set of conditions visible in this lemma. That the conditions on U_m and V_m can be satisfied for all possible parameter combinations is best proofed by giving examples on how to construct these operators. Such examples are provided by the lemmas 4.1.4 and 4.1.5, found below. □

To numerically find parameters combinations that satisfy the condition in the above lemma, a little computer program was used (an analytical solution may also be possible—I haven't tried it). An important data structure in the program is the vector $\vec{s}$ whose entries are sets of parameter combinations $\{a_0, a_1, b_0, b_1, \beta_0, \beta_1, \gamma_0, \gamma_1\}$ and some extra information like the corresponding disturbance and postestimation deviation. The entries are sorted corresponding to the disturbance, for example s_0 may contain an entry whose disturbance is between 0 and 0.1, s_1 an entry whose disturbance is between 0.1 and 0.2, and so on (the precise ranges can be set using certain parameters in the program). What the program does is to sample over all parameter combinations. For each combination, it calculates the disturbance s and postestimation v' deviation. It then looks up the corresponding entry in the vector $\vec{s}$. If the entry contains a parameter combination with a lower postestimation deviation than that of the new parameter combination, it does nothing, otherwise it updates the entry with the new parameters. The output of the program is the vector $\vec{s}$ which contains the best parameter combinations for each disturbance range. Of course, the parameter combinations are not perfect since, due to the sampling process, only a subset of all possible combinations can be checked.

The results of a run of the program are displayed in table 4.1. What catches the eye are the frequent "jumps" in the parameters. For example in the fifth line the value of a_0 is 0.040404, one line below it is 0.989899, and on the sixth line it jumps back to something small: 0.10101. The explanation is that there are several different parameter combinations possible for an optimal disturbance-deviation pair. Which one gets chosen by the program depends on the way the samples are selected and on numerical inaccuracies.

Figure 4.2 displays the disturbance-estimation graph generated

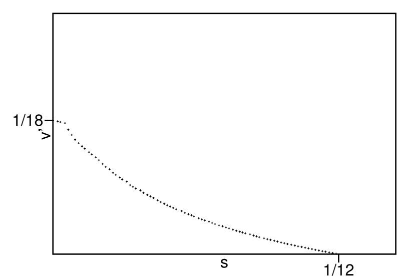

Figure 4.2: Disturbance s and postestimation deviation v' plotted from the data in the table 4.1.

from the values in the table that we just discussed. A peculiar feature of that graph is the kink close to where the disturbance is 0 and where the deviation is 1/18. An explanation for it may be that the grid used for sampling over the parameters was not fine enough and that, therefore, some values that would make the graph smoother were missed. On the other hand, the kink may be natural. But let us not worry about too much about this peculiarity and turn to something else. We see that the SMP2 procedure is better than the equivalent SMMP2 procedure which is based solely on minimal measurements. With the definition of minimal measurements, it follows that the measurement operators $M_m = U_m D_m V_m$ described by the values in table 4.1 cannot all be positive. To construct the operators from these values we must find unitary operators U_m and V_m that satisfy the conditions in lemma 4.1.3. The following two lemmas show us one way to accomplish that.

Lemma 4.1.4. *If* $m \in \{0,1\}$, $a_m \in [0,1]$, $b_m \in [0,1]$, *and* $\gamma_m \in [0,1]$, *then the operator*

$$V_m = \sqrt{1-\gamma_m} e^{i \arg x_m} |0\rangle \langle 0| + \sqrt{\gamma_m} e^{-i\pi/2} |0\rangle \langle 1| + \sqrt{\gamma_m} |1\rangle \langle 0| + \sqrt{1-\gamma_m} e^{i(\pi/2 - \arg x_m)} |1\rangle \langle 1|$$

with

$$x_0 = a_0^2 - b_0^2, x_1 = -(a_1^2 - b_1^2)$$

is unitary and satisfies the conditions $|\langle 0| V_m |1\rangle|^2 = \gamma_m$ *and*

$$\arg\big((a_0^2 - b_0^2) \langle 0| V_0^\dagger |0\rangle \langle 0| V_0 |1\rangle\big) = \arg\big(-(a_1^2 - b_1^2) \langle 0| V_1^\dagger |0\rangle \langle 0| V_1 |1\rangle\big).$$

Proof. That V_m is unitary and satisfies the given conditions follows by insertion and straight forward computation. □

Lemma 4.1.5. *If* $m \in \{0,1\}$, $\beta_m \in [0,1]$, *and* $|\xi_m\rangle$ *is a vector in qubit state space, then the operator*

$$U_m = \sqrt{1-\beta_m} e^{i(\arg\langle 1|\xi_m\rangle - \pi/2)} |0\rangle \langle 0| + \sqrt{\beta_m} e^{i(\pi/2 + \arg\langle 0|\xi_m\rangle)} |0\rangle \langle 1| + \sqrt{\beta_m} e^{-i\arg\langle 0|\xi_m\rangle} |1\rangle \langle 0| + \sqrt{1-\beta_m} e^{-i\arg\langle 1|\xi_m\rangle} |1\rangle \langle 1|$$

is unitary and satisfies the conditions $|\langle 0|\chi_m\rangle|^2 = \beta_m$ *and*

$$\big(-\arg\langle 0|\chi_m\rangle + \arg\langle 1|\chi_m\rangle + \arg\langle 0|\xi_m\rangle - \arg\langle 1|\xi_m\rangle\big) \mod 2\pi = 0,$$

where $|\chi_m\rangle = U_m^\dagger |1\rangle$.

Proof. Insertion and straight forward computation proves that V_m satisfies the given conditions and is unitary. □

v'	s	a_0	a_1	b_0	b_1	β_0	β_1	γ_0	γ_1
1.54838e-18	0.0555556	0	0	1	1	0.333333	0.666667	0	1
0.00139737	0.0551754	0.0808081	0	1	0.99673	0.666667	0.333333	0.989899	0.010101
0.00215049	0.0549066	0.10101	0	1	0.994885	0.333333	0.656566	0.020202	0.979798
0.00349971	0.0544498	1	0.991341	0.131313	0	0.323232	0.656566	0.030303	0.969697
0.00446202	0.0517227	0.040404	0.141774	0.989899	0.999183	0.646465	0.333333	0.939394	0.0606061
0.00549089	0.0495278	0.989899	0.99673	0.0808081	0.141774	0.363636	0.676768	0.0808081	0.919192
0.0064734	0.0475727	0.10101	0.141774	0.989899	0.994885	0.373737	0.636364	0.121212	0.878788
0.00750355	0.0460888	0.989899	0.992627	0.121212	0.141774	0.616162	0.383838	0.838384	0.161616
0.00850195	0.044759	0.141414	0.141774	0.989899	0.989951	0.373737	0.626263	0.161616	0.838384
0.00928516	0.0438229	0.151515	0.141774	0.989899	0.988455	0.40404	0.59596	0.222222	0.777778
0.0105085	0.0423433	0.171717	0.141774	0.989899	0.985146	0.606061	0.40404	0.777778	0.222222
0.0113412	0.0414573	0.989899	0.983332	0.181818	0.141774	0.424242	0.575758	0.282828	0.717172
0.012496	0.0402149	0.141414	0.19999	0.979798	0.989951	0.585859	0.40404	0.737374	0.262626
0.0133911	0.0390587	0.151515	0.19999	0.979798	0.988455	0.474747	0.525253	0.424242	0.575758
0.0144829	0.0374564	0.171717	0.19999	0.979798	0.985146	0.434343	0.575758	0.313131	0.686869
0.0153783	0.0362475	0.979798	0.983332	0.181818	0.19999	0.505051	0.494949	0.515152	0.484848
0.0165067	0.0352366	0.979798	0.979381	0.20202	0.19999	0.414141	0.575758	0.30303	0.69697
0.017517	0.0337978	0.212121	0.19999	0.979798	0.977243	0.464646	0.525253	0.424242	0.575758
0.018332	0.0328835	0.979798	0.974996	0.222222	0.19999	0.494949	0.505051	0.484848	0.515152
0.0195069	0.0318674	0.969697	0.981411	0.191919	0.244311	0.494949	0.505051	0.484848	0.515152
0.0202288	0.0310401	0.969697	0.979381	0.20202	0.244311	0.494949	0.505051	0.484848	0.515152
0.021517	0.0301419	0.262626	0.19999	0.979798	0.964898	0.545455	0.474747	0.575758	0.424242
0.0225171	0.0287073	0.969697	0.972639	0.232323	0.244311	0.494949	0.505051	0.484848	0.515152
0.0233166	0.0279753	0.969697	0.97017	0.242424	0.244311	0.494949	0.505051	0.484848	0.515152
0.0241325	0.0272643	0.969697	0.96759	0.252525	0.244311	0.494949	0.505051	0.484848	0.515152
0.0255212	0.0264784	0.232323	0.281382	0.959596	0.972639	0.474747	0.545455	0.424242	0.575758
0.0263944	0.0254688	0.959596	0.97017	0.242424	0.281382	0.494949	0.505051	0.494949	0.505051
0.0275261	0.0246474	0.292929	0.244311	0.969697	0.956134	0.484848	0.515152	0.474747	0.525253
0.0284208	0.023986	0.969697	0.952981	0.30303	0.244311	0.505051	0.494949	0.505051	0.494949
0.0293135	0.0233793	0.969697	0.94971	0.313131	0.244311	0.505051	0.494949	0.505051	0.494949
0.030527	0.0223473	0.292929	0.281382	0.959596	0.956134	0.474747	0.525253	0.454545	0.545455
0.0314369	0.021624	0.959596	0.952981	0.30303	0.281382	0.505051	0.494949	0.505051	0.494949
0.0324952	0.0209193	0.949495	0.959171	0.282828	0.313782	0.505051	0.494949	0.505051	0.494949
0.0333437	0.0203281	0.949495	0.956134	0.292929	0.313782	0.505051	0.494949	0.505051	0.494949
0.0342025	0.0197532	0.949495	0.952981	0.30303	0.313782	0.505051	0.494949	0.505051	0.494949
0.0350707	0.0191939	0.949495	0.94971	0.313131	0.313782	0.505051	0.494949	0.505051	0.494949
0.0359475	0.0186493	0.949495	0.94632	0.323232	0.313782	0.505051	0.494949	0.505051	0.494949
0.0375333	0.0179311	0.313131	0.34284	0.939394	0.94971	0.464646	0.535354	0.444444	0.555556
0.0384789	0.0171416	0.939394	0.94632	0.323232	0.34284	0.505051	0.494949	0.505051	0.494949
0.039524	0.0166043	0.949495	0.931541	0.363636	0.313782	0.505051	0.494949	0.505051	0.494949
0.0404318	0.016123	0.949495	0.927535	0.373737	0.313782	0.494949	0.505051	0.494949	0.505051
0.0411097	0.0156504	0.939394	0.935421	0.353535	0.34284	0.505051	0.494949	0.505051	0.494949
0.0425383	0.0149318	0.343434	0.369344	0.929293	0.939177	0.484848	0.515152	0.474747	0.525253
0.0434199	0.0144234	0.929293	0.935421	0.353535	0.369344	0.505051	0.494949	0.505051	0.494949
0.0442923	0.01397	0.929293	0.931541	0.363636	0.369344	0.494949	0.505051	0.494949	0.505051
0.045535	0.0134828	0.353535	0.39381	0.919192	0.935421	0.525253	0.474747	0.535354	0.464646
0.0464321	0.012923	0.919192	0.931541	0.363636	0.39381	0.505051	0.494949	0.505051	0.494949
0.0472921	0.0124984	0.919192	0.927535	0.373737	0.39381	0.505051	0.494949	0.505051	0.494949
0.0484353	0.0120001	0.909091	0.931541	0.363636	0.416598	0.494949	0.505051	0.494949	0.505051
0.0494912	0.0115801	0.89899	0.935421	0.353535	0.437969	0.494949	0.505051	0.494949	0.505051
0.050467	0.0110879	0.434343	0.369344	0.929293	0.900747	0.505051	0.494949	0.505051	0.494949
0.0513517	0.0107122	0.929293	0.895806	0.444444	0.369344	0.505051	0.494949	0.505051	0.494949
0.0524834	0.0101622	0.434343	0.39381	0.919192	0.900747	0.505051	0.494949	0.505051	0.494949
0.0535165	0.00970234	0.909091	0.905549	0.424242	0.416598	0.505051	0.494949	0.505051	0.494949
0.054465	0.00931245	0.89899	0.910213	0.414141	0.437969	0.494949	0.505051	0.494949	0.505051
0.055294	0.00896829	0.424242	0.437969	0.89899	0.905549	0.505051	0.494949	0.505051	0.494949
0.0561215	0.00863262	0.89899	0.900747	0.434343	0.437969	0.505051	0.494949	0.505051	0.494949
0.0574924	0.00828312	0.515152	0.369344	0.929293	0.857099	0.505051	0.494949	0.505051	0.494949
0.0585325	0.00771178	0.878788	0.905549	0.424242	0.477213	0.505051	0.494949	0.505051	0.494949
0.0593793	0.00736789	0.888889	0.890724	0.454545	0.458123	0.505051	0.494949	0.505051	0.494949
0.0601783	0.00706943	0.888889	0.885496	0.464646	0.458123	0.505051	0.494949	0.505051	0.494949
0.0615494	0.00658263	0.444444	0.495362	0.868687	0.895806	0.494949	0.505051	0.494949	0.505051
0.062549	0.00621581	0.888889	0.868922	0.494949	0.458123	0.505051	0.494949	0.505051	0.494949
0.0634647	0.00593152	0.434343	0.52922	0.848485	0.900747	0.494949	0.505051	0.494949	0.505051
0.0645606	0.00552961	0.484848	0.495362	0.868687	0.874598	0.484848	0.515152	0.484848	0.515152
0.0654822	0.00519854	0.515152	0.477213	0.878788	0.857099	0.505051	0.494949	0.505051	0.494949
0.0665656	0.00483184	0.858586	0.868922	0.494949	0.51267	0.505051	0.494949	0.505051	0.494949
0.067493	0.00452911	0.868687	0.850946	0.525253	0.495362	0.505051	0.494949	0.505051	0.494949
0.0684353	0.00422357	0.848485	0.86309	0.505051	0.52922	0.505051	0.494949	0.505051	0.494949
0.0695236	0.00388039	0.505051	0.54508	0.838384	0.86309	0.494949	0.505051	0.494949	0.505051
0.0705477	0.00356369	0.505051	0.56031	0.828283	0.86309	0.494949	0.505051	0.494949	0.505051
0.0715119	0.00327061	0.818182	0.86309	0.505051	0.57496	0.494949	0.505051	0.494949	0.505051
0.0725362	0.00298261	0.616162	0.477213	0.878788	0.78762	0.505051	0.494949	0.505051	0.494949
0.0735655	0.00266655	0.868687	0.78762	0.616162	0.495362	0.505051	0.494949	0.505051	0.494949
0.0745218	0.00237155	0.838384	0.810413	0.585859	0.54508	0.505051	0.494949	0.505051	0.494949
0.0755558	0.00208845	0.505051	0.640837	0.767677	0.86309	0.494949	0.505051	0.494949	0.505051
0.0765556	0.00180022	0.525253	0.640837	0.767677	0.850946	0.494949	0.505051	0.494949	0.505051
0.0775623	0.00152044	0.848485	0.754269	0.656566	0.52922	0.505051	0.494949	0.505051	0.494949
0.0785764	0.00124303	0.787879	0.803014	0.59596	0.61583	0.484848	0.515152	0.484848	0.515152
0.0795346	0.000980127	0.606061	0.628539	0.777778	0.795418	0.494949	0.505051	0.494949	0.505051
0.0805697	0.000707077	0.646465	0.61583	0.787879	0.762944	0.505051	0.494949	0.505051	0.494949
0.0815659	0.000448765	0.606061	0.686349	0.727273	0.795418	0.494949	0.505051	0.494949	0.505051
0.082579	0.000191073	0.474747	0.824641	0.565657	0.880122	0.494949	0.505051	0.494949	0.505051
0.0832689	-5.59042e-05	0.010101	1	0	0.999949	0.494949	0.505051	0.494949	0.505051

Table 4.1: Approximation of optimal parameters for an SMP2 procedure according to lemma 4.1.3.

An interesting set of measurement operators are those that account for disturbance 0 and deviation 1/18. Their approximate parameters are listed in the first line of table 4.1. By using the approximations $0.333333 \approx \frac{1}{3}$, $0.666667 \approx \frac{2}{3}$ and the above lemmas it follows that $M_m = U_m D_m V_m$ ($m \in \{0,1\}$) can be composed of

$$\begin{aligned}
&V_0 = -\left|0\right\rangle\left\langle 0\right| - i\left|1\right\rangle\left\langle 1\right|, \; V_1 = \left|0\right\rangle\left\langle 0\right| + i\left|1\right\rangle\left\langle 1\right|, \\
&U_0 = -\sqrt{2/3}\left|0\right\rangle\left\langle 0\right| + i\sqrt{1/3}\left|0\right\rangle\left\langle 1\right| + \sqrt{1/3}\left|1\right\rangle\left\langle 0\right| + i\sqrt{2/3}\left|1\right\rangle\left\langle 1\right|, \\
&U_1 = \sqrt{1/3}\left|0\right\rangle\left\langle 0\right| + i\sqrt{2/3}\left|0\right\rangle\left\langle 1\right| + \sqrt{2/3}\left|1\right\rangle\left\langle 0\right| - i\sqrt{1/3}\left|1\right\rangle\left\langle 1\right|, \\
&D_0 = \left|1\right\rangle\left\langle 1\right|, \; D_1 = \left|1\right\rangle\left\langle 1\right|.
\end{aligned}$$

The corresponding guesses are $g'_0 = 2/3$, $g'_1 = 1/3$. Another interesting set of operators are those that account for deviation 0 and disturbance 1/12. By inference from the values of the last line of table 4.1 if follows with the above two lemmas that they can be chosen to be

$$M_0 = 0, \; M_1 = \sqrt{1/2}\Big((1+i)\big(\left|0\right\rangle\left\langle 0\right| - \left|0\right\rangle\left\langle 1\right|\big) + (1-i)\big(\left|1\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right|\big)\Big).$$

The guess corresponding to M_1 is $g'_1(1) = 1/2$ (specifying a guess for M_0 is pointless since the measurement outcome 0 is unobtainable if $M_0 = 0$). Why this rather peculiar set of operators accounts for disturbance 0 needs further investigation. From the discussion of the SMMP2 procedure we recall that another, less peculiar, set of operators with disturbance 0 and deviation 1/12 is $\{M_m = \frac{1}{2}\mathbb{1} : m \in \{0,1\}\}$. This set, of course, could also be used together with the SMP2 procedure.

4.1.3 Postestimation with Three or More Outcomes

We see that the SMP2 procedure is much better than the SMP1 procedure (this is to be expected since the SMP1 procedure does not rely on measurement results). This brings up an interesting question: Is a procedure with many outcomes always better than one with less outcomes? A first step towards the answer to that question could involve the observation of a procedure with three outcomes. However, if such a procedure is, given optimal parameters, not better than one with two outcomes, one has to be careful not to draw a wrong conclusion. For example, there could be some a that procedures with an odd number of outcomes are at a disadvantage when compared to procedures with an even number of outcomes.

4.2 Uniform Sequential Postestimation

In this section we treat postestimation procedures that comprise a series of measurements, mostly as a preparation for tracking. Let us define a general such procedure:

USP Procedure.
Full name: Uniform Sequential Postestimation Procedure.
Parameters:
- A set of measurement operators, $\{M_0, M_1, \ldots, M_{Z-1}\}$, $Z \geq 1$.
- The number of measurements, N.
- The distance in time between measurements, τ.
- A guess function, g'_1.

Steps:
1. A sequence of N measurements, separated in time by τ, is applied to the state ψ of the qubit under observation. The readout of the measurements is stored in the vector $\vec{m}$. The state of ψ after the measurements is denoted by ψ'.
2. Using the readout $\vec{m}$, the state of $|c'_1|^2$ is estimated with the guess $g'_1(\vec{m})$.

Under certain side conditions, it is possible to specify the best guess in a closed form:

Lemma 4.2.1. *The optimal guess for the USP procedure under the side condition that*
- *the initial state vector, $|\psi_0\rangle$, of the qubit under observation is completely unknown,*
- *the knowledge about the operator U that mediates the time evolution between two consecutive measurements is given by the probability density $\varrho(U)$,*

is

$$g'_{1,\vec{m}} = \frac{\int dU\, \varrho(U)\, \langle 1|\, \tilde{E}_{\vec{m}}(U)\, |1\rangle}{\int dU\, \varrho(U)\, \langle 0|\, E_{\vec{m}}(U)\, |0\rangle + \int dU\, \varrho(U)\, \langle 1|\, E_{\vec{m}}(U)\, |1\rangle},$$

where $\vec{m}$ is the vector containing the measurement results (the measurement operators are denoted by $M_0, M_1, \ldots, M_{Z-1}$,$Z \geq 1$) and

$$M_{\vec{m}}(U) = M_{m_{Z-1}} U M_{m_{Z-2}} U \cdots U M_{m_0}, \tag{4.2.1}$$

$$E_{\vec{m}}(U) = M^{\dagger}_{\vec{m}}(U) M_{\vec{m}}(U), \tag{4.2.2}$$

$$\tilde{E}_{\vec{m}}(U) = M_{\vec{m}}(U) M^{\dagger}_{\vec{m}}(U). \tag{4.2.3}$$

Proof. The first step is the calculation of the postestimation deviation of the tracking procedure. It turns out to be

$$v' = \int d\psi_0 \varrho(\psi_0) \int dU \varrho(U) \sum_{\vec{m}'} p(\vec{m}'|\psi_0, U) \left(g'_{1,\vec{m}'} - |c_1(\vec{m}', \psi_0, U)|^2\right)^2.$$

Because the disturbance doesn't depend on the guess, all that is left to do is to find the set of guesses that minimizes the postestimation deviation. Since $v\prime$ has only one extremum, which is a minimum, it suffices to take the derivative of v', set it to zero, and solve the resulting equation for $g'_{1,\vec{m}}$. The result is the expression found in the conjecture. □

In may be interesting to also find the best set of measurement operators under the side conditions mentioned in the above lemma. This, however, is probably a non trivial task.

4.3 Uniform Tracking

We will now take the USP procedure, build a corresponding tracking procedure from it, and apply it to the task of tracking the system that we have discussed in section 3.2.

UT Procedure.
Full name: Uniform Tracking Procedure.
Parameters:

- The set of measurement operators, $\{M_0, M_1, \ldots, M_{Z-1}\}$, $Z \geq 1$.
- The number of measurements, N.
- The distance in time between measurements, τ.
- A guess function, g_1

Steps: The state $\psi(t)$ of the qubit under observation is measured at times t_0, $t_0 + 1\tau$, $t_0 + 2\tau$, $\ldots$, $t_0 + (N-1)\tau$, where t_0 is the time when tracking starts. In parallel, the evolution of the parameter $|c_1(t)|^2 = |\langle 1|\psi(t)\rangle|^2$ is continuously estimated using the guess $g_1(\vec{m}_{N'})$, where $\vec{m}_{N'}$ is the vector containing the readout of the first N' measurements.

Due to the similarities between both procedures, the optimal guess for the USP procedure is also the optimal guess for the UT procedure. In order to simulate the UT procedure applied to a system with the Hamiltonian from section 3.2, it would be nice if that guess could be rewritten into a form that doesn't contain any integrals (they would take up far too much computation time). This can be done. But before doing so, we must learn how to rewrite the time evolution operator based on the Hamiltonian from section 3.2 into a certain form:

Lemma 4.3.1. *The time evolution operator* $e^{iH_I\tau/\hbar}$ *with* $H_I = \hbar D(|1\rangle\langle 0| + |0\rangle\langle 1|)$ *is identical to*

$$\cos(\varphi)\mathbb{1} + i\sin(\varphi)\sigma_x,$$

where σ_x *is the x-Pauli operator [NC02] and* $\theta = -D\tau$.

Proof. The identity follows from the decomposition of the exponential function using Pauli operators as explained in the chapter about linear algebra in the Nielsen Chuang [NC02]. ☐

What follows is the promised guess:

Theorem 4.3.1. *The optimal guess for the UT procedure under the side condition that*

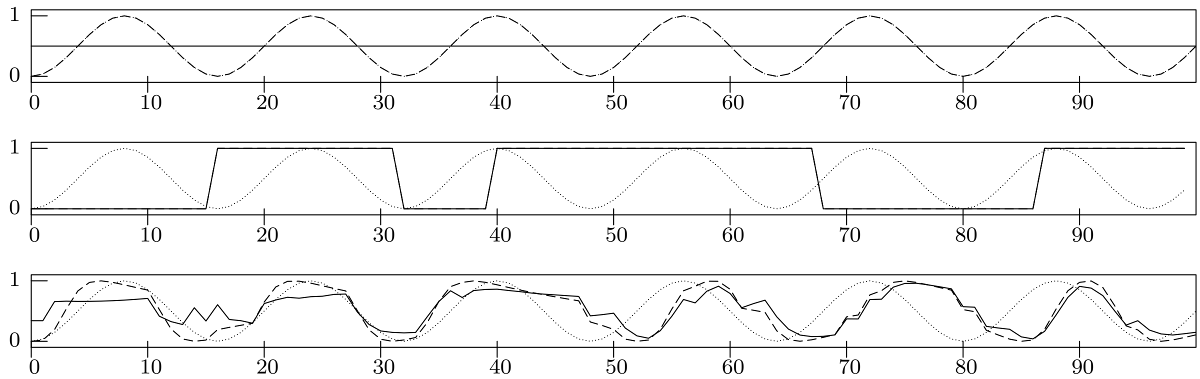

Figure 4.3: Simulation of the USP procedure applied to a qubit with a Hamiltonian of the form $H_{\mathrm{I}} = \hbar D(|1\rangle\langle 0| + |0\rangle\langle 1|)$, $D > 0$. The employed measurement operators are $M_0 = \sqrt{p_0}\,|0\rangle\langle 0| + \sqrt{p_1}\,|p_1\rangle\langle p_1|$ and $M_1 = \sqrt{1-p_0}\,|0\rangle\langle 0| + \sqrt{1-p_1}\,|p_1\rangle\langle p_1|$ with $\bar{p} = \frac{1}{2}(p_0 + p_1) = \frac{1}{2}$ and $\Delta p = p_1 - p_0 = 0$ (top plot), $\Delta p = 1$ (middle plot), $\Delta p = 0.32$ (bottom plot). The dotted curve corresponds to the undisturbed evolution (i.e. without measurements) of the parameter $|c_1|^2$ of the observed qubit, the dashed curve corresponds to actual evolution of $|c_1|^2$, and the solid curve corresponds to the guess g_1 which was calculated according to theorem 4.3.1. Whole numbers on the horizontal axis denote the position of measurements.

- *the initial state vector, $|\psi_0\rangle$, of the qubit under observation is completely unknown,*
- *the unitary operator mediating the time evolution between measurements is known to be $U(\varphi) = \cos(\varphi)\mathbb{1} + i\sin(\varphi)\sigma_x$ but φ is completely unknown (i.e. it's probability distribution function is $\varrho(\varphi) = 1/(2\pi)$,*

is

$$g_1(\vec{m}_{N'}) = \frac{\overline{\overline{E}}(\vec{m}_{N'})_{11}}{\overline{E}(\vec{m}_{N'})_{00} + \overline{E}(\vec{m}_{N'})_{11}},$$

where

$$\begin{aligned}\overline{E}(\vec{m}_{N'})_{jk} =& \frac{1}{2\pi}\sum_{p=0}^{1}\sum_{n=0}^{N_0'}(a_{pjn}^{(N')})^* \\ & \sum_{q=\lfloor (n+1)/2\rfloor}^{\lceil (n+N_0'-1)/2\rceil} a_{pk(2q-n)}^{(N')} I_{sc2}(q, N_0'-q), \\ \overline{\overline{E}}(\vec{m}_{N'})_{jj} =& \frac{1}{2\pi}\sum_{n=0}^{N_0'} a_{jj(2n)}^{(N')} I_{sc2}(n, N_0'-n)\end{aligned}$$

with $N'_0 = N' - 1$ *and*

$$I_{sc2}(n,m) = \frac{2}{(n+m)!}\Gamma(n+\frac{1}{2})\Gamma(m+\frac{1}{2}),$$

$$a^{(N)}_{jkn} = \begin{cases} \langle j|\, M_{m_0}\,|k\rangle = \langle j|\, M_{m_{N'_0}}\,|k\rangle & \text{for } N=1,\\ \sum_{p=0}^{1}\big(M_{m_{N'_0}}\big)_{jp}\big(a^{(N'-1)}_{pk(n-1)}\delta(n\geq 1)+ & \\ \qquad\qquad ia^{(N'-1)}_{(1-p)kn}\delta(n\leq N'_0-1)\big) & \text{for } N\geq 2. \end{cases}$$

Proof. The best guess for the USP procedure (see lemma 4.2.1) is, due to the similarity of the procedures, also the best guess for the UT procedure. Starting with it, the next step is to eliminate the integrals over $\langle 1|\,\tilde{E}_{\vec{m}_{N'}}\,|1\rangle$, $\langle 0|\,E_{\vec{m}_{N'}}\,|0\rangle$, and $\langle 1|\,E_{\vec{m}_{N'}}\,|1\rangle$. This is done by expressing the integrands as bivariate polynomials in $\sin(\varphi)$ and $\cos(\varphi)$, whose coefficients are given by the recursive relation $a^{(N)}_{jkn}$. These polynomials can then be integrated and the results are the above expressions for $\overline{E}(\vec{m}_{N'})_{jk}$ and $\overline{\tilde{E}}(\vec{m}_{N'})_{jj}$. Thus, the conjecture is true. □

Figure 4.3 shows the results of simulations of the UT procedure using minimal measurements with the following measurement operators (the parameterization is the same as that used in chapter 3)

$$M_0 = \sqrt{p_0}\,|0\rangle\,\langle 0| + \sqrt{p_1}\,|1\rangle\,\langle 1|\,,$$
$$M_1 = \sqrt{1-p_0}\,|0\rangle\,\langle 0| + \sqrt{1-p_1}\,|1\rangle\,\langle 1|$$

For the top plot of figure 4.3, the parameters p_0 and p_1 where chosen so that M_0 and M_1 are proportional to the unit operator. Therefore, the observed qubit is not disturbed at all. On the other hand, no information about the evolution of $|c_1|^2$ is gained and, thus, the guess stays at 1/2 throughout.

In the middle plot of figure 4.3, the measurement operators are projective. Consequently, the evolution of the qubit is heavily disturbed and, because the information gain is high, the guess $g_1(t)$ can be very close to $|c_1|^2$ at all times. Note that it actually looks as if $g_1(t)$ coincides with $|c_1(t)|^2$ at all times. This, however, is not the case because $|c_1|^2$ evolves in between measurements (recall the discussion of the UPMT procedure in chapter 2). This evolution was simply not recorded during the simulation.

For the bottom plot of figure 4.3, the parameters of the measurement operators where chosen so that the disturbance and deviation nate kept in balance. What we see is interesting. In the beginning, the guess $g_1(t)$ is not very good. Then, later, it slowly approaches $|c_1|^2$. It looks as if the guess is learning. Of course, since it knows the form of the Hamiltonian and since it takes into account all previous measurement results, it is not surprising that the optimal guess is as good as shown.

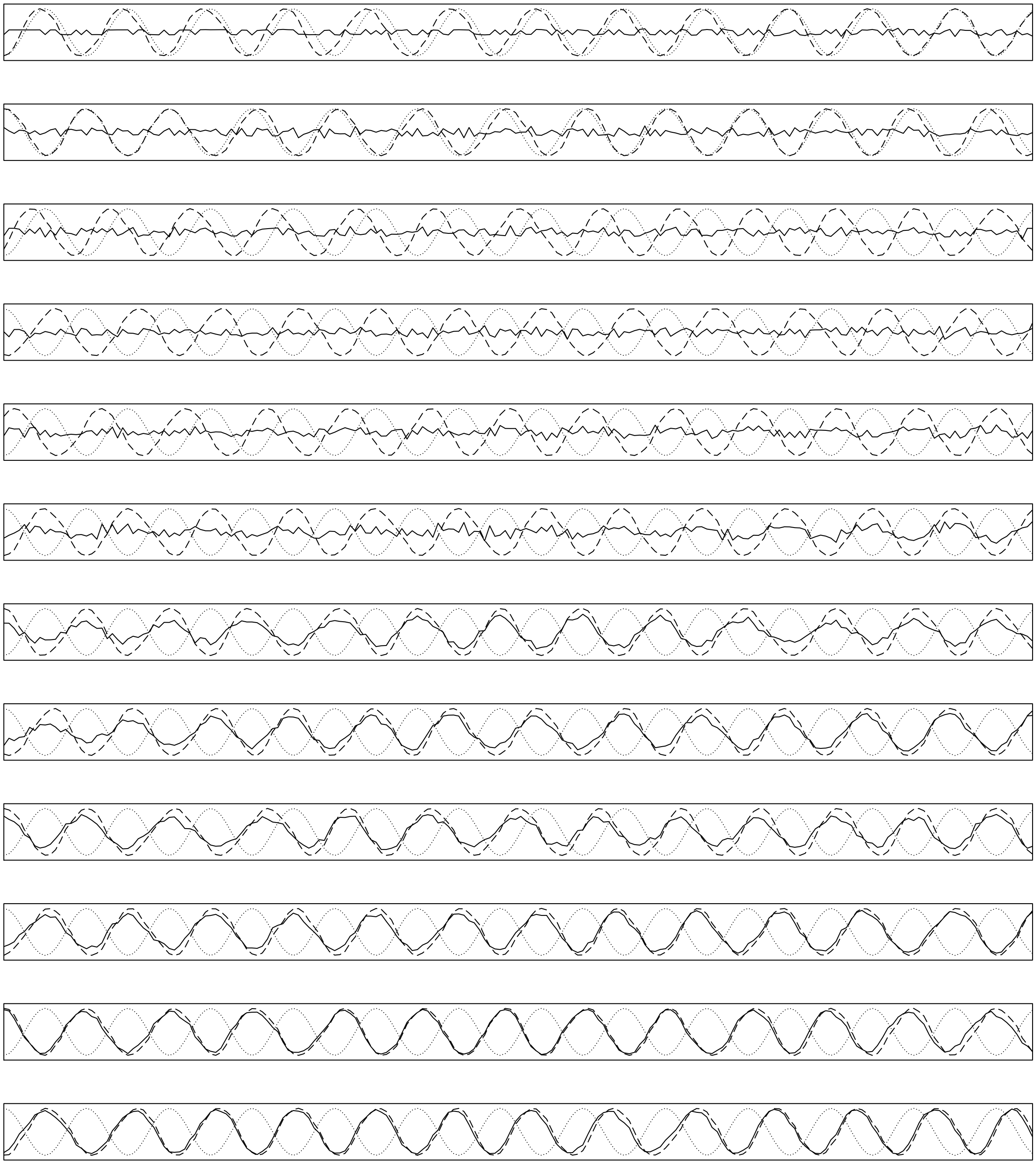

Figure 4.4: Evolution of a qubit under the influence of the UT procedure using measurement operators $M_0 = \sqrt{p_0}\,|0\rangle\langle 0| + \sqrt{p_1}\,|p_1\rangle\langle p_1|$ and $M_1 = \sqrt{1-p_0}\,|0\rangle\langle 0| + \sqrt{1-p_1}\,|p_1\rangle\langle p_1|$ with $\Delta p = p_1 - p_0 = 0$ and $\bar{p} = \frac{1}{2}(p_0 + p_1) = \frac{1}{2}$ (simulated with Trasim). The dotted curve corresponds to the undisturbed evolution (i.e. without measurements) of the parameter $|c_1|^2$ of the observed qubit, the dashed curve corresponds to actual evolution of $|c_1|^2$, and the solid curve corresponds to the guess g_1 which was calculated according to theorem 4.3.1. Each line corresponds to 200 measurements.

The high quality of the guess gives the above UT procedure a large advantage over other procedures: It is possible to observe the evolution of $|c_1|^2$ while only mildly disturbing the system by individual measurements. A plot that nicely illustrates this feature is displayed in figure 4.4. It must, however, be pointed out that, although individual measurements cause little disturbance, the sum of all these measurements causes considerable disturbance. It is to be assumed that the offset α, the phase φ, and the amplitude β of the original time evolution (i.e. before tracking starts)

$$|c_1(t)|^2 = \alpha + \beta \cos(\varphi + \omega t)$$

become eventually almost completely "destroyed".

We conclude this section with some remarks on optimal measurement operators. The above choice of $M_m = \sqrt{\alpha_0^m} |0\rangle \langle 0| + \sqrt{\alpha_1^m} |1\rangle \langle 1|$ ($\alpha_j^0 = p_j$, $\alpha_j^1 = 1 - p_j$), possibly, is not as good as it seems on first sight. When gauging the quality in terms of disturbance and deviation, the following operators may be an equal, if not better, alternative:

$$M_m = \sqrt{\alpha_0^m} |q_0\rangle \langle q_0| + \sqrt{\alpha_1^m} |q_1\rangle \langle q_1| ,$$

where $|q_0\rangle$ and $|q_1\rangle$ are the eigenstates of the Hamiltonian of the system. They would cause $|c_1(t)|^2$ to "dive" variably quick towards a stationary value. Consequently, a guess a with a stationary value is sufficient to achieve a very low deviation. The disturbance is probably in a similar range as when using the original measurement operators (recall that, as explained above, these operators eventually cause a considerable phase shift). The disadvantage of these alternative parameters is obvious: The dynamics of the system is suppressed.

The choice of optimal parameters deserves further investigation. Especially interesting is the case where no a priory information about the Hamiltonian of the observed qubit is available. For example in the case that not even the minimum expectable frequency is known, it may be necessary to make the temporal distance between measurements an adaptive quantity. Solutions may be found in the theory of discrete Fourier transformations.

5 Conclusion and Outlook

Let us reflect on some of the postestimation and tracking procedures that we have discussed in this thesis. The basic procedures discussed in the beginning of chapter 2 (namely the SPMP procedure and the UPMT procedure) look quite primitive when compared to the procedures that we discussed later. Anyone having experience with projective measurements would not have expected anything else. They were presented mainly for completeness and to introduce general principles. What turned out to be interesting, though, were the frequency determination and tracking procedures presented at the end of that chapter. As mentioned there, it might be fruitful to solve the problem of finding the frequency determination procedure with the quickest convergence. A related problem, whose solution probably has a wider application, is to find the quickest procedure for determining the entire Hamiltonian of a system. One could try to use general measurements for the solution of this problem, although, at first sight, it looks as if projective measurements are just the right tool.

In chapter 2, several procedures employing minimal measurements were discussed. We learned that the optimal parameters for the SMMP procedure are somewhat disappointing since the corresponding guess is always 1/2. The N-series tracking procedure did not get much discussion. Other tracking procedures, most notably the UT procedure that we discussed in chapter 3, seem to be much more powerful because they take into account *all* previous measurement results instead of just those from the last N-series and, therefore, make the use of awkward filters unnecessary.

In the first part of chapter 4, we saw that postestimation with minimal measurements is, fortunately, not “as far as it gets”: The optimal parameters of the SMP2 procedure account for a guess that is not just 1/2. It may be interesting to find out what are the best parameters for the case of the USP procedure which is based on sequential general measurements. This would likely also lead to the best parameters for the UT procedure. Further, one could, instead of using discrete measurements, use so called *continuous measurements*, for the purpose of tracking. This approach has been discussed by Audretsch et. al. [AKM01]. An open question seems to be, what are the best parameters in terms of disturbance and deviation for this kind of procedure?

It should be noted that, although used throughout this thesis, disturbance and deviation may not always be the best choice of measures when gauging the quality of estimation and tracking procedures. What measures are a good choice depends on the features of the observed qubit that are desired to be seen (for example, for the FD procedure, where only the oscillation frequency is of interest, the concept of disturbance and deviation is of not much use). The approach of finding the optimal estimation and tracking procedures by minimizing the disturbance and deviation can, of course, be extended to other quality criteria as well. Such criteria might for example be the *fidelity*, a measure for the overlap of two state vectors. In fact these criteria have been used to gauge the quality of procedures for estimating the entire state vector of a qubit by a single measurement [Ban01] [ADK03]. It may be promising to use them for finding a good procedure for tracking the entire state vector of a qubit by sequential measurements.

Another research proposal is the extension of tracking and estimation to multi qubit systems. This, however, is a, conceptually, nontrivial task since it is hard to visualize the state of these systems. To avoid that problem, one could only concentrate on features of multi qubit systems that can for example be represented as points in a two dimensional plane. Maybe this could even lead to some interesting applications which currently seem to be missing from the theory of tracking $|c_1|^2$.

Tools for Computation in Qubit State Space

Lemma .0.2. *If $\varrho(\psi)$ is a uniform probability density of qubit states and f is an integrable function on qubit state space, then*

$$\int d\psi\, \varrho(\psi) f(|\psi\rangle) = \frac{1}{4\pi}\int_0^\pi d\vartheta\, \sin\vartheta \int_0^{2\pi} d\varphi\, f\big(\cos\frac{\vartheta}{2}\,|0\rangle + e^{i\varphi}\sin\frac{\vartheta}{2}\,|1\rangle\big).$$

Proof. According to its definition in chapter 1, $\varrho(\psi)$ is homogeneously distributed over the surface of the Bloch sphere (i.e. it is everywhere the same on the Bloch sphere). Therefore, the integral of f over all states, weighted with $\varrho(\psi)$, is proportional to an integral of f over the surface of the Bloch sphere:

$$\int d\psi \varrho(\psi)\, f(|\psi\rangle) = C\int_0^\pi d\vartheta\, \sin\vartheta \int_0^{2\pi} d\varphi\, \varrho(\vartheta,\varphi) f\big(\cos\tfrac{\vartheta}{2}\,|0\rangle + e^{i\varphi}\sin\tfrac{\vartheta}{2}\,|1\rangle\big), \quad (.0.1)$$

where the factor C is defined by the normalization of $\varrho(\psi)$:

$$\int d\psi\, \varrho(\psi) = 1 \overset{(.0.1)}{\Leftrightarrow} C\int_0^\pi d\vartheta\, \sin\vartheta \int_0^{2\pi} d\varphi = 1 \Leftrightarrow C = \tfrac{1}{4\pi}.$$

□

Lemma .0.3. *If $\varrho(\psi)$ is a uniform probability density of qubit states, if $|\chi\rangle$ is a qubit state, and if $f : [0,1] \to \mathbb{C}$ is a function that is integrable on $[0,1]$, then*

$$\int d\psi\, \varrho(\psi) f(|\langle\chi|\psi\rangle|^2) = \int_0^1 d\alpha\, f(\alpha).$$

Proof. By imagining the Bloch sphere, it is easily seen that, because $\varrho(\psi)$ is uniform, the value of $\int d\psi\, \varrho(\psi) f(|\langle\chi|\psi\rangle|^2)$ must be independent of the direction of $|\chi\rangle$. Therefore, we can substitute $|\chi\rangle$ by $|1\rangle$:

$$\int d\psi\, \varrho(\psi) f(|\langle\chi|\psi\rangle|^2) = \int d\psi\, \varrho(\psi) f(|\langle 1|\psi\rangle|^2).$$

According to lemma .0.2 the right side of this expression is equal to

$$\tfrac{1}{4\pi}\int_0^\pi d\vartheta\, \sin\vartheta \int_0^{2\pi} d\varphi\, f(\sin^2\tfrac{\vartheta}{2}),$$

which, by standard rules of integration, together with the substitution $\alpha \equiv \sin^2(\vartheta/2)$ evaluates to $\int_0^1 d\alpha\, f(\alpha)$. □

Trasim

Trasim is a computer program that was used for the creation of many graphs in this thesis (see chapter on how to obtain it). Its name is a concatenation of the first three letters of each of the words *tracking* and *simulation*. With this knowledge it's easy to guess that Trasim's purpose is the simulation of tracking procedures, though, during the creation of this thesis, Trasim has also been (ab)used to simulate just sequences of measurements (in this case the "guess-output" explained below has been ignored). Many details of the program are explained in the documentation that is included in the Trasim software package (the file README in the main directory of the package is a good starting point). Let us therefore look just at its most important features:

Input: Input to Trasim is encoded in an input file that the user of the program specifies as a command line parameter. It is primarily comprised of

- the names of the output files,
- the Hamiltonian H of the simulated system,
- the initial state $|\psi_{\text{init}}\rangle$ of the system,
- the tracking procedure to be simulated (e.g. frequency tracking) and its parameters (e.g. the temporal distance between measurements),
- the type of guess to be used,
- the measurement operators to be used,
- the duration of the tracking procedure.

Output: Trasim's output consists of two files. One contains miscellaneous data, like for example the time that the simulation took on the computer. The other file contains data describing the simulated temporal evolution of the system and the guess.

Main steps: How and when the program reads and writes input and output data shall not interest us for this short overview. Let us instead have a look at the most important steps during the simulation of a tracking procedure. After the system's state vector $|\psi\rangle$ is initialized with $|\psi_{\text{init}}\rangle$, the following steps are repeated for the duration of the tracking procedure:

1. A measurement of $|\psi\rangle$ is simulated. It's outcome is determined with the help of a pseudo random number generator and $|\psi\rangle$ is transformed into the corresponding post measurement state.
2. The guess for $|c_1|^2 = |\langle 1|\psi\rangle|^2$ is calculated. It depends on the outcomes of the preceding measurement or measurements.

3. The time evolution of the system's state vector $|\psi\rangle$ is simulated by sequential application of operators $e^{iH\tau_i/\hbar}$, where the τ_i sum up to the temporal distance between the last and the next measurement. This sequential application is useful when the user wants to get a picture of the time evolution of $|c_1|^2$ in between measurements.

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